IL

NUOVO CIMENTO

ORGANO DELLA SOCIETÀ ITALIANA DI FISICA

SOTTO GLI AUSPICI DEL CONSIGLIO NAZIONALE DELLE RICERCHE

VOL. XII, N. 4 — *Serie decima* — 16 Maggio 1959

A Statistical Theory of Lattice Damage in Solids Irradiated by High-Energy Particles.

L. I. PÁL and G. NÉMETH

Central Research Institute for Physics, Hungarian Academy of Sciences - Budapest

(ricevuto il 1° Novembre 1958)

Summary. — The distribution function for the number of lattice displacements in solids irradiated by high-energy particles has been determined by means of exact methods of the theory of probability. The generating function of the distribution function yields equations determining the average number of displacements and their dispersion. These equations have been solved on the assumption that the primary bombarding particles are neutrons and that the collision phenomena between a displaced atom and an atom vibrating in the lattice may be regarded as collisions between perfectly rigid spheres. It has been found that the variance of the number of lattice displacements is larger than the variance of Poisson's distribution for the case of irradiation by fission neutrons. If the thickness of the irradiated sample is small compared against the mean free path of the neutrons, then the relative standard deviation is given by

$$\frac{\mathcal{D}(t)}{\mathcal{N}_1(t)} \sim \sqrt{\left[\frac{2.167}{Q_s b} + \frac{D_\gamma^2(t)}{S_1(t)} - 1\right]\frac{1}{S_1(t)}}\cdot \\ \cdot\left\{1 + \frac{0.055}{(1-\alpha)[2.167 + Q_s b(D_\gamma^2(t)/S_1(t) - 1)]}\frac{E_d}{E_f} + \dots\right\},$$

where $S_1(t)$ denotes the average number of bombarding neutrons penetrating the unit surface of the specimen during an exposure t and $D_\gamma^2(t) = S_2(t) - S_1^2(t)$ is its variance. Q_s is the macroscopic collision cross-section for the material to be irradiated, b is the thickness of the specimen, E_d is the displacement energy in eV and $E_f = 0.5\cdot 10^6$ eV. Finally $\alpha = ((M-1)/(M+1))^2$ where M is the mass number of the specimen nucleus. $\mathcal{N}_1(t)$ denotes the average number of lattice displacements during an exposure time t.

1. – Introduction.

In a solid irradiated by high-energy particles (*e.g.* fast neutrons) individual atoms of the crystal lattice are displaced from their equilibrium positions and after a longer or shorter path—during which they may displace further atoms from their cell positions—they may come to rest at some « illegal » point of the crystal lattice as interstitial atoms. Thus the crystal lattice will show the presence of « vacancies » and interstitial atoms. Ejected atoms during their displacement lose their energy through ionization, inelastic collision (excitation of lattice vibration) and elastic collision. The major portion of the energy thus transmitted to the crystal lattice will be dissipated in exciting lattice vibrations, while the minor portion will engender the ejections of atoms.

In the interest of characterizing in a quantitative manner the lattice damage caused by irradiation it would be necessary to calculate carefully the average number of displaced atoms and possibly its variance. Although no reliable methods are available for determining experimentally the average number of damaged lattice positions, it can be safely stated that theoretical estimates and experimental results hitherto show a significant divergence. Theoretical considerations give a value for the average number of displaced atoms which is $(5 \div 6)$ times greater than that computed from experimental data. While it may be assumed that the real cause of this divergence must be found in the elementary acts of displacement, it may seem useful to elaborate a general theoretical method to discuss the complex process by means of the theory of probability.

Computing methods hitherto published ([1-3]) do not allow the investigation of radiation damage by means of the theory of probability. A consistent (at least for a given model) method is needed also in order to estimate the standard deviation of the number of displaced atoms in samples irradiated under identical conditions, thus separating the fluctuations due to the statistical character of the phenomenon from real physical variations.

By means of the method developed by one of the authors ([4]) the distribution function of the number of displaced atoms may be defined and—by using the generating function of this distribution function—semiinvariants characterizing the distribution (average value, dispersion, etc.) may be computed. Our method considers the statistical nature of the elementary displacement act so that

(¹) F. Seitz and W. A. Harrison: *Phys. Rev.*, **98**, 1530 (1955); F. Seitz and J. S. Koehler: *Solid State Physics* (New York, 1956), vol. 2, pp. 381 ff.

(²) W. S. Snyder and J. Neufeld: *Phys. Rev.*, **97**, 1936 (1955).

(³) W. S. Snyder and J. Neufeld: *Phys. Rev.*, **103**, 862 (1956).

(⁴) L. Pál: *Energia és Atomtechnika*, **10**, 255 (1957).

the trend seen from the studies of A. E. FEIN [5] is being consequently applied in our considerations.

In the following chapters a consistent statistical theory of radiation damage is being developed and its relations to earlier methods discussed.

2. – Derivation of the basic equations.

Let ν_{E_0} denote the actual number of atoms ejected directly or indirectly by a bombarding particle having the energy E_0: Determine the probability of $\nu_{E_0} = n$, *i.e.* let us determine the distribution function

$$\mathcal{P}(\nu_{E_0} = n) = P_n(E_0) \,. \tag{2.1}$$

In order to define the distribution function $P_n(E_0)$ the following initial assumptions are needed. Let the irradiated material be of semiinfinite dimensions in half-space. The empty half-space and the material to be irradiated are bounded by the infinite plane (y, z) (see Fig. 1). In order to simplify our discussion let us assume that neither the atoms ejected by the bombarding particles nor the bombarding particles themselves can enter the empty half-space leaving the material to be irradiated. Our considerations—of course—retain their validity also for solids of finite dimensions, as the small mean free path of the displaced atoms results in concentrating their effect to a relatively small volume element. No consequences may be drawn as to the spatial distribution of lattice defects. The distribution function $P_n(E_0)$ gives information only upon the probability of generating n displaced atoms by a bombarding particle having energy E_0 in the total volume of the sample exposed to radiation.

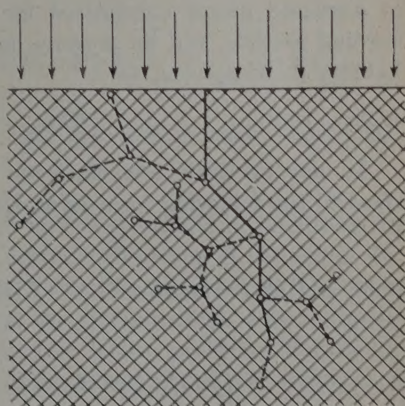

Fig. 1. – Lattice displacements in solids irradiated by high-energy particles. The full line refers to the path of the primary particle while the dotted lines refer to the paths of the displaced atoms.

In our computations recombination effects due to occupation of free vacant places by diffusion of ejected atoms are neglected, and only the possible capture of the ejecting atom by the lattice position of the ejected atom is being con-

([5]) A. E. FEIN: *Phys. Rev.*, **109**, 1076 (1958).

sidered. It can be expected that the value thus obtained for the average number of displaced atoms will be larger than the actual value. For the sake of simplicity let us suppose that the material to be irradiated consists of only one species of atoms.

Denote by $w_p(E_0, E)\,\mathrm{d}E$ the probability that the energy E_0 of the bombarding particle will be reduced by a single collision to a value within the interval $(E, E+\mathrm{d}E)$. For the time being no limitations shall be imposed as to the way of this energy reduction. The atom hit by the bombarding particle gains an energy E_0-E during the collision. This energy may be dissipated *e.g.* in ionization, in the excitation of lattice vibrations, or—in a favourable case—in the ejection of the atom from its equilibrium position. Denote by $K(E_0-E)$ the probability of this ejection being actually obtained if an atom receives the energy E_0-E from the bombarding particle. Denote further by $q(E_0-E, E')\,\mathrm{d}E'$ the probability of the kinetic energy of the ejected atom falling into the interval $(E', E'+\mathrm{d}E')$ supposing that prior to its ejection it has gained an energy E_0-E from its colliding partner. The ejected atom having an energy E' may obviously displace further atoms. Let $\varrho_{E'}$ be the number of displaced atoms engendered by the atom directly ejected by the bombarding particle, and its progeny during its path. Denote by $p_m(E')$ the probability (*) of $\varrho_{E'}=m$, *i.e.*

$$\mathcal{P}(\varrho_{E'}=m)=p_m(E')\,. \tag{2.2}$$

It may be easily seen that the distribution function $P_n(E_0)$ may be derived in the following form using the probability factors hitherto introduced:

$$\begin{aligned} P_n(E_0) = {} & \int_0^{E_0} w_p(E_0, E)\{1-K(E_0-E)\}\,P_n(E)\,\mathrm{d}E\,+ \\ & + \int_0^{E_0} w_p(E_0, E)\,K(E_0-E)\left\{\sum_{m=0}^{n} P_{n-m}(E)\int_0^{E_0-E} q(E_0-E, E')\,p_m(E')\,\mathrm{d}E'\right\}\mathrm{d}E\,. \end{aligned} \tag{2.3}$$

The distribution function $p_m(E)$ is represented by the solution of the following integral equation

$$\begin{aligned} p_m(E) = {} & \int_0^{E} w_s(E, E')\{1-K(E-E')\}\,p_m(E')\,\mathrm{d}E'\,+ \\ & + \int_0^{E} w_s(E, E')\,K(E-E')\left\{\sum_{m'=0}^{m} p_{m-m'}(E')\int_0^{E-E'} q(E-E', E'')\,p_{m'}(E'')\,\mathrm{d}E''\right\}\mathrm{d}E', \end{aligned} \tag{2.4}$$

(*) $p_m(E)=\delta_{1m}$, if E is lower than the minimum limit energy for atom displacement.

where $w_s(E, E')\,\mathrm{d}E'$ denotes the probability of the energy of an ejected atom being reduced during collision with another atom to within the interval $(E', E'+\mathrm{d}E')$, supposing that its energy prior to collision has been given by E. As the spatial distribution of displaced atoms is not being investigated, anisotropy of the ejection will remain unconsidered.

Equations (2.3) and (2.4) can be written after introduction of the generating functions

$$(2.5)\qquad G(E_0, x) = \sum_{n=0}^{\infty} e^{nx} P_n(E_0), \qquad g(E, x) = \sum_{m=0}^{\infty} e^{mx} p_m(E)\,,$$

in the following form

$$(2.6)\qquad G(E_0, x) = \int_0^{E_0} w_p(E_0, E)\,G(E, x)\left\{1 - K(E_0 - E) + \right.$$
$$\left. + K(E_0 - E)\int_0^{E_0 - E} q(E_0 - E, E')\,g(E', x)\,\mathrm{d}E'\right\}\mathrm{d}E\,,$$

and

$$(2.7)\qquad g(E, x) = \int_0^{E} w_s(E, E')\,g(E', x)\left\{1 - K(E - E') + \right.$$
$$\left. + K(E - E')\int_0^{E-E'} q(E - E', E'')\,g(E'', x)\,\mathrm{d}E''\right\}\mathrm{d}E'.$$

The number of atoms ejected is reduced by the possibility of the displacing atoms being captured with a given probability to the place of the atoms displaced. Therefore equations (2.4) and (2.7) cannot be regarded as fully exact equations. Let $l(E')$ be the probability that the ejecting atom, with an energy E' after collision is being captured by the free lattice position. Further let $p_m^0(E)$ be the probability that an atom having an energy E will engender m displacements if the possibility of capture is being taken into account. It may be simply proven that the generating function $g_0(E, x) = \sum_{m=0}^{\infty} e^{mx} p_m^0(E)$ satisfies the equation

$$(2.7')\qquad g_0(E, x) = \int_0^{E} w_s(E, E')[1 - K(E - E')]\,g_0(E', x)\,\mathrm{d}E' +$$
$$+ \int_0^{E} w_s(E, E')K(E - E')\{[1 - l(E')]\,g_0(E', x) + l(E')\}\int_0^{E-E'} q(E - E', E'')\,g_0(E'', x)\mathrm{d}E''\,\mathrm{d}E'.$$

By use of (2.7′) and knowing the functions w_s, K, q and l all other important characteristics of the distribution function $p_m^0(E)$ can be computed. In our

further discussion calculations will be based on equations (2.6) and (2.7), as only crude qualitative assumptions can be made as to the function $l(E')$.

Let us suppose that every surface element of the sample to be irradiated is exposed to radiation having uniform intensity and energy spectrum. Let us denote by $R(t, E_1, ..., E_k)\,\mathrm{d}E_1 ... \mathrm{d}E_2$ the probability that during an exposure of the duration t the unity surface of the solid will be penetrated by k bombarding particles with an energy within the intervals $(E_1, E_1+\mathrm{d}E_1)$, .., $(E_k, E_k+\mathrm{d}E_k)$. If denoting now by $\mathcal{L}_n(t)$ the probability that during an exposure time t through the unit surface n displaced atoms will be encountered, then one may write the following relation:

$$\mathcal{L}_n(t) = \sum_{k=0}^{\infty} \int_0^{\infty} ... \int_0^{\infty} R(t, E_1, ..., E_k) \sum_{n_1+...+n_k=n} P_{n_1}(E_1) ... P_{n_k}(E_k)\,\mathrm{d}E_1 ... \mathrm{d}E_k\,. \tag{2.8}$$

Without infringing our principles of generality we may suppose that

$$R(t, E_1, ..., E_k) = R_k(t) \prod_{i=1}^{k} h(E_i)\,. \tag{2.8'}$$

Let us introduce now the generating functions

$$H(t, x) = \sum_{k=0}^{\infty} e^{kx} R_k(t) \quad \text{and} \quad F(t, x) = \sum_{n=0}^{\infty} e^{nx} \mathcal{L}_n(t)\,.$$

Through their application the following relation is being obtained

$$F(t, x) = H\left\{t, \ln\int_0^{\infty} h(E_0)\,G(E_0, x)\,\mathrm{d}E_0\right\}, \tag{2.9}$$

from which all necessary quantities may be easily derived.

3. – The average number of displaced atoms and its dispersion.

3·1. – The average number of displaced atoms may be computed from (2.9) by use of the relation

$$N_1(t) = \left[\frac{\partial F(t, x)}{\partial x}\right]_{x=0}. \tag{3.1}$$

Let us introduce the following notations:

$$M_1(E_0) = \left[\frac{\partial G(E_0, x)}{\partial x}\right]_{x=0}, \qquad S_1(t) = \left[\frac{\partial H(t, x)}{\partial x}\right]_{x=0}, \tag{3.1'}$$

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$M_1(E_0)$ is the average number of displaced atoms produced by a bombarding particle with energy E_0. The quantity $S_1(t)$ defines the average number of bombarding particles penetrating the unit surface of the specimen during an exposure time t. Finally the number of atoms displaced by an irradiation through the unit surface and for an exposure time t will be given by

$$N_1(t) = S_1(t)\int_0^\infty h(E_0)\, M_1(E_0)\, \mathrm{d}E_0\,, \tag{3.2}$$

where $M_1(E_0)$ may be computed from the equation

$$M_1(E_0) = \int_0^{E_0} w_p(E_0, E)\left\{M_1(E) + K(E_0 - E)\int_0^{E_0-E} q(E_0 - E, E')\, m_1(E')\, \mathrm{d}E'\right\}\mathrm{d}E\,. \tag{3.3}$$

The function $m_1(E)$ appearing in equation (3.3) may be derived from the generating function (2.7) by use of the rule $m_1(E) = [\partial g(E, x)/\partial x]_{x=0}$. The defining equation has the following form

$$m_1(E) = \int_0^{E} w_s(E, E')\left\{m_1(E') + K(E - E')\int_0^{E-E'} q(E - E', E'')\, m_1(E'')\, \mathrm{d}E''\right\}\mathrm{d}E'\,. \tag{3.4}$$

3·2. – Equation (2.9) will yield without difficulties the dispersion of the number of displaced atoms, which could only be computed in a rather intricate manner without the knowledge of the distribution function. Denote by $D^2(t)$ the variance of the number of atoms displaced by an irradiation through the unit surface during an exposure time t. This can be defined obviously through the following relation:

$$D^2(t) = \left[\frac{\partial^2 \ln F(t, x)}{\partial x^2}\right]_{x=0}. \tag{3.5}$$

Let us introduce the following notations:

$$M_2(E_0) = \left[\frac{\partial^2 G(E_0, x)}{\partial x^2}\right]_{x=0}, \qquad S_2(E_0) = \left[\frac{\partial^2 H(t, x)}{\partial x^2}\right]_{x=0}, \tag{3.6}$$

as well as

$$D_d^2 = \int_0^\infty h(E_0)\, M_2(E_0)\, \mathrm{d}E_0 - \left\{\int_0^\infty h(E_0)\, M(E_0)\, \mathrm{d}E_0\right\}^2. \tag{3.7}$$

and

$$D_r^2(t) = S_2(t) - S_1^2(t) . \tag{3.7'}$$

From (3.5) it can be obtained that

$$D^2(t) = S_1(t) D_d^2 + D_r^2(t) \left\{ \int_0^\infty h(E_0) M_1(E_0) \,\mathrm{d}E_0 \right\}^2 . \tag{3.8}$$

In order to determine (3.8) one has to compute the momenta $M_1(E_0)$, $M_2(E_0)$ and to know the functions $h(E_0)$, $S_1(t)$, $S_2(t)$. It may be easily seen that $M_2(E_0)$ will satisfy the equation

$$\begin{aligned} M_2(E_0) = & \int_0^{E_0} w_p(E_0, E) \left\{ M_2(E) + K(E_0 - E) \int_0^{E_0 - E} q(E_0 - E, E') m_2(E') \,\mathrm{d}E' \right\} \mathrm{d}E + \\ & + 2 \int_0^{E_0} w_p(E_0, E) K(E_0 - E) M_1(E) \int_0^{E_0 - E} q(E_0 - E, E') m_1(E') \,\mathrm{d}E' \,\mathrm{d}E . \end{aligned} \tag{3.9}$$

For $m_2(E)$ one obtains the following equation as based on (2.7):

$$\begin{aligned} m_2(E) = & \int_0^{E} w_s(E, E') \left\{ m_2(E') + K(E - E') \int_0^{E - E'} q(E - E', E'') m_2(E'') \,\mathrm{d}E'' \right\} \mathrm{d}E' + \\ & + 2 \int_0^{E} w_s(E, E') K(E - E') m_1(E') \int_0^{E - E'} q(E - E', E'') m_1(E'') \,\mathrm{d}E'' \,\mathrm{d}E' . \end{aligned} \tag{3.10}$$

Now the actual task is to solve equations (3.3) and (3.4) as well as (3.9) and (3.10). For this one has to know the density functions w_s, w_p and q as well as the displacement probability value K. It is a problem of quantum theory to define these functions. In view of the complex character of this problem no valuable results have been hitherto obtained. As can be seen from the following, the functions may be defined and computations carried out for certain simple cases.

4. – The average number of lattice defects in solids irradiated by neutrons and its deviation.

This part has the purpose of illustrating the general theory developed in Parts **2** and **3** on the SEITZ-HARRISON version of the hard-sphere collision model.

If the primary particles are neutrons, then (at least for the case of not excessively large energies) one has:

$$w_p(E_0, E) = \begin{cases} (1-\alpha)^{-1} E_0^{-1} & \text{if} \quad E_0 \geqslant E \geqslant \alpha E_0 , \\ 0 & \text{if} \quad \alpha E_0 > E , \end{cases} \tag{4.1}$$

where

$$\alpha = \left(\frac{M_2 - M_1}{M_2 + M_1}\right)^2 . \tag{4.2}$$

In relation (4.2) M_2 denotes the mass of the atom at rest and M_1 the mass of the bombarding neutron. Atoms displaced by the neutrons may eject further atoms.

It is known that collisions between ionized atoms may be regarded—below certain energy limits—as a collision between ideally rigid spheres, because of the shielding effect of the electron shell of atoms. In the following it is being supposed that collisions between atoms displaced by the primary bombarding particles may be regarded as collision phenomena of this type. Energy transfer during collision of identical atoms will be then defined by

$$w_s(E, E') = E^{-1} \qquad (E \geqslant E' \geqslant 0). \tag{4.3}$$

For simplicity's sake let us suppose that collisions lead always to displacement, if the energy transferred to the atom (considered to be at rest prior to collision) is above the displacement energy level E_d. For this case it may be easily proved that

$$K(E) = \begin{cases} 1 & \text{if} \quad E \geqslant E_d , \\ 0 & \text{if} \quad E < E_d , \end{cases} \tag{4.4}$$

and

$$q(E, E') = \delta(E - E' - E_d) . \tag{4.5}$$

a) Let us now compute the average value

$$N_1(t) = S_1(t) \int_0^{\infty} h(E_0) M_1(E_0) \, dE_0 , \tag{4.6}$$

by considering the conditions enumerated above. First of all we write down the equation defining the function $M_1(E_0)$. On the basis of (4.1), (4.3), (4.4), and (4.5) our equation (3.3) and (3.4) after minor transformations may be

written down in the following form

$$(4.7)\qquad (1-\alpha)E_0 M_1(E_0) = \int\limits_{\alpha E_0}^{E_0} M_1(E)\,\mathrm{d}E + \int\limits_0^{(1-\alpha)E_0 - E_d} m_1(E)\,\mathrm{d}E\,,$$

$$M_1(E_0) = 0 \qquad \text{if} \qquad 0 \leqslant E_0 \leqslant (1-\alpha)^{-1}E_d$$

and

$$(4.8)\qquad Em_1(E) = \int\limits_0^E m_1(E')\,\mathrm{d}E' + \int\limits_0^{E-E_d} m_1(E')\,\mathrm{d}E'\,,$$

$$m_1(E) = 1 \qquad \text{if} \qquad 0 \leqslant E \leqslant E_d\,.$$

Equation (4.8) is exactly identical to the equations derived by SEITZ and HARRISON (1) as well as by SNYDER and NEUFELD (2). However, equation (4.7) differs from equation (62) as given in the paper by SNYDER and NEUFELD. If however it is being supposed that the bombarding neutron will suffer only one collision in the material exposed to irradiation, then the equation (4.7) becomes exactly identical with that of SNYDER and NEUFELD. This assumption however forces us to modify equation (4.6) as, with large probability, the neutrons penetrating the sample will collide only once. (The sample may be regarded as being of infinite thickness as compared against the mean free path of the displaced atoms, but it will be very thin so far as the mean free paths of the bombarding neutrons are concerned.) Denote by $Q_s(E_0)$ the macroscopic scattering cross-section of the neutrons having energy E_0 in a material of thickness b exposed to radiation. In place of (4.6) the following expression should be used:

$$(4.8')\qquad \mathcal{N}_1(t) = S_1(t)\int\limits_0^\infty h(E_0)\mathcal{M}_1(E_0)\,\mathrm{d}E_0\,,$$

where $\mathcal{M}_1(E_0)$ represents the solution of the equation

$$(4.8'')\qquad \mathcal{M}_1(E_0) = \frac{Q_s(E_0)b}{(1-\alpha)E_0}\int\limits_0^{(1-\alpha)E_0-E_d} m_1(E)\,\mathrm{d}E\,.$$

b) Our expression (3.8) for the variance will assume a particularly simple form if one assumes that $S_2(t) - S_1^2(t) = S_1(t)$. In this case one has

$$(4.9)\qquad D^2(t) = S_1(t)\int\limits_0^\infty h(E_0)\,M_2(E_0)\,\mathrm{d}E_0\,,$$

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and the relative variance will be given by

$$\frac{D^2(t)}{N_1(t)} = \frac{\int\limits_0^\infty h(E_0)\, M_2(E_0)\, \mathrm{d}E_0}{\int\limits_0^\infty h(E_0)\, M_1(E_0)\, \mathrm{d}E_0}\,. \tag{4.10}$$

Now the actual problem is to determine $M_2(E_0)$. Let us transform equations (3.9) and (3.10) by considering relations (4.1), (4.3), (4.4), and (4.5). Short calculations will yield

$$(1-\alpha)E_0 M_2(E_0) = \int\limits_{\alpha E_0}^{E_0} M_2(E)\,\mathrm{d}E + \int\limits_0^{(1-\alpha)E_0 - E_d} m_2(E)\,\mathrm{d}E + 2\int\limits_0^{(1-\alpha)E_0 - E_d} M_1(E_0 - E_d - E)\, m_1(E)\,\mathrm{d}E\,, \tag{4.11}$$

and

$$E m_2(E) = \int\limits_0^{E} m_2(E')\,\mathrm{d}E' + \int\limits_0^{E-E_d} m_2(E')\,\mathrm{d}E' + 2\int\limits_0^{E-E_d} m_1(E - E_d - E')\, m_1(E')\,\mathrm{d}E'\,. \tag{4.12}$$

Supposing again that the bombarding neutrons in a material of thickness b will suffer only one single collision, equation (4.11) may be substituted by equation

$$\mathcal{M}_2(E_0) = \frac{Q_s(E_0)b}{(1-\alpha)E_0} \int\limits_0^{(1-\alpha)E_0 - E_d} m_2(E)\,\mathrm{d}E\,. \tag{4.12'}$$

Variance will be given for this case—instead of (4.9)—by the following formula:

$$\mathcal{D}^2(t) = S_1(t) \int\limits_0^\infty h(E_0)\mathcal{M}_2(E_0)\,\mathrm{d}E_0\,. \tag{4.13}$$

Generally the suitable expression is to be obtained from (3.8) by substituting $\mathcal{M}_1(E_0)$ and $\mathcal{M}_2(E_0)$ for $M_1(E_0)$ and $M_2(E_0)$.

c) In order to utilize the expressions (4.6) and (3.8) one has to compute the first and second moments of the generating function $g(E, x)$. It appears to be purposeful therefore to investigate the equation

$$E g(E, x) = \int\limits_0^{E} g(E', x)\,\mathrm{d}E' - \int\limits_0^{E-E_d} g(E', x)\,[1 - g(E - E_d - E')]\,\mathrm{d}E'\,, \tag{4.14}$$

to be derived by considering the conditions (4.1), (4.3), (4.4), and (4.5) and define its solution. Let

$$U(z, x) = \int^{\infty} e^{-zE} g(E, x)\, dE . \tag{4.15}$$

A short computation procedure will yield the following result

$$U(z, x) = z^{-1}\{1 - (1 - e^{-x}) \exp[E_1(zE_d)]\}^{-1} , \tag{4.16}$$

where

$$E_1(zE_d) = \int\limits_{zE_d}^{\infty} \frac{e^{-y}}{y}\, dy .$$

This will easily yield the distribution function (*) although it is much more important for us to define the momenta $m_1(E)$ and $m_2(E)$. Let us introduce the following notations

$$\mu_1(z) = \int\limits_0^{\infty} \exp[-zE] m_1(E)\, dE \qquad \text{and} \qquad \mu_2(z) = \int\limits_0^{\infty} \exp[-zE] m_2(E)\, dE . \tag{4.16'}$$

Expressions (4.16) will give us through the relations

$$\mu_1(z) = \left[\frac{\partial U(z, x)}{\partial x}\right]_{x=0} \qquad \text{and} \qquad \mu_2(z) = \left[\frac{\partial^2 U(z, x)}{\partial x^2}\right]_{x=0},$$

the following formulae:

$$\mu_1(z) = z^{-1} \exp[E_1(zE_d)] \tag{4.17}$$

and

$$\mu_2(z) = 2z\mu_1^2(z) - \mu_1(z) . \tag{4.18}$$

Utilizing (4.17) and (4.18) asymptotical expressions may be obtained for $m_1(E)$ and $m_2(E)$ for values of $E \gg E_d$. By simple considerations the asymptotical

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(*) If $E \gg E_d$ with a simple calculation we obtain the relation

$$p_m(E) \sim (1 - e^{C})^{m-1} e^{C - E/E_d} L_{m-1}\left[\frac{E}{E_d(1 - e^{-C})}\right],$$

where $C = 0.577$ and $L_m(y)$ is the Laguerre polynomial.

relations

$$m_1(E) \sim e^{-C}\frac{E}{E_d} + e^{-C} + \dots, \qquad (E \gg E_d), \tag{4.19}$$

and

$$m_2(E) \sim e^{-2C}\frac{E^2}{E_d^2} + e^{-C}(4e^{-C}-1)\frac{E}{E_d} + e^{-C}(3e^{-C}-1) + \dots, \qquad (E \gg E_d), \tag{4.20}$$

will be obtained. The constant C figuring in formulae (4.19) and (4.20) is the so-called Euler constant ($C = 0.577$). The expression (4.19) coincides with Seitz and Koehler's equation (29.5) in (¹).

The variance $d^2(E) = m_2(E) - m_1^2(E)$ will be given by

$$d^2(E) \sim e^{-C}(2e^{-C}-1)\frac{E}{E_d} + \dots, \qquad (E \gg E_d). \tag{4.21}$$

Leibfried (⁶) has obtained a slightly different result because in formulating his equations he followed the model of Kinchin and Pease (⁷), while we chose the model of Seitz and Harrison (¹). Some difference between the Leibfried formula and (4.21) appears to be present in the principal number, also its value is of less significance. The latter has the value $0.075\,E/E_d$ according to Leibfried while $0.069\,E/E_d$ results from (4.21).

On the basis of $m_1(E)$ and $m_2(E)$ and by utilizing equations (4.7) and (4.11) momenta $M_1(E_0)$ and $M_2(E_0)$ may be computed.

Equation (4.7) will yield after the substitutions $E_0 = E_d e^{u_0}$ and $M_1(E_d e^{u_0}) = T(u_0)$ for the value of the Laplace transform $\vartheta(z) = \int_0^\infty \exp[-u_0 z]T(u_0)\,du_0$ the following relation

$$\vartheta(z) = \vartheta_1(z)\,\vartheta_2(z)\,,$$

where

$$\vartheta_1(z) = \frac{z(1-\alpha)^{z+1}}{\alpha^{z+1} + (1-\alpha)(z+1) - 1}, \tag{4.22}$$

$$\vartheta_2(z) = z^{-1}\int_0^\infty e^{-zy} m_1[E_d(e^y - 1)]\,dy\,. \tag{4.23}$$

In order to determine the form of $T(u_0)$ for the case of large u_0 values, all poles of the function $\vartheta(z)$ must be known for which the real part of the function is not negative. One factor of $\vartheta(z)$, *i.e.* $\vartheta_1(z)$ is already well known for the

(⁶) G. Leibfried, *Nukleonik*, **1**, 57 (1958).
(⁷) G. H. Kinchin and R. S. Pease: *Rep. Progr. Phys.*, **18**, 1 (1955).

theory of slowing down of neutrons. Poles with non-negative real parts of $\vartheta_2(z)$ have been determined by G. NÉMETH [8]. In possession of our knowledge concerning $\vartheta_1(z)$ and $\vartheta_2(z)$ the following presentation is being arrived at:

$$\vartheta_1(z) = \frac{e^{-c}}{z-1} - \frac{1-e^{-c}}{\xi z} + \vartheta_3(z) \;(^*),$$

which will yield for $M_1(E_0)$ the following formula:

$$M_1(E_0) \sim e^{-c}\frac{E_0}{E_d} - \frac{1-e^{-c}}{\xi} + \dots, \qquad (E_0 \gg E_d), \tag{4.24}$$

$$\xi = 1 + \frac{\alpha}{1-\alpha}\ln\alpha .$$

(4.24) does not agree with the expression (62) given by SNYDER and NEUFELD [2]. The reason for this is that whereas it was assumed by SNYDER and NEUFELD that the bombarding neutron underwent only a single collision in the irradiated sample, we considered the complete slowing down of the bombarding neutron past the energy level E_d and determined the average displaced atoms in the slowing down process. On the other hand formula (4.27) holds for so thin a sample where $Q_s b \ll 1$ and thus the probability that the bombarding neutron would suffer in the sample more than one collision is of the order of magnitude $(Q_s b)^k$ with $k > 1$.

Taking (4.11) and repeating the procedure applied in the case of (4.7) the following asymptotical formula will be obtained for $M_2(E_0)$:

$$M_2(E_0) \sim e^{-2c}\frac{E_0^2}{E_d^2} + e^{-c}[2e^{-c}(1+1/\xi) - (1+2/\xi)]\frac{E_0}{E_d} + \dots, \quad (E_0 \gg E_d). \tag{4.25}$$

The principal member of the variance expression will be given by

$$D^2(E_0) \sim e^{-c}(2e^{-c}-1)\frac{E_0}{E_d} + \dots . \tag{4.26}$$

It can be seen that consideration of the number of collisions suffered by the primary bombarding neutron will not influence the relative variance of lattice defects (at least not its principal member corresponding to large values of E_0).

If the sample to be irradiated may be considered as being thin when compared against the mean free path of the bombarding neutrons then it will be

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([8]) G. NÉMETH: private communication.

(*) $\vartheta_3(z)$ has not poles with not-negative real parts.

highly probable that each neutron will collide only once during penetration of the sample. In this case equations (4.8″) and (4.12′) will yield the momenta $\mathcal{M}_1(E_0)$ and $\mathcal{M}_2(E_0)$. A short computation will result in the following relations

$$\mathcal{M}_1(E_0) \sim \frac{1}{2} Q_s b(1-\alpha)\, e^{-c} \frac{E_0}{E_d} + \dots, \qquad (E_0 \gg E_d), \tag{4.27}$$

$$\mathcal{M}_2(E_0) \sim \frac{1}{3} Q_s b(1-\alpha)^2 e^{-2c} \frac{E_0}{E_d} + \frac{1}{2} Q_s b(1-\alpha)\, e^{-c} (2e^{-c}-1) \frac{E_0}{E_d} + \dots, \qquad (E_0 \gg E_d). \tag{4.28}$$

The variance will be given by the formula

$$\mathcal{D}^2(E_0) \sim \left(\frac{1}{3} - \frac{1}{4} Q_s b\right)(1-\alpha)^2 Q_s b\, e^{-2c} \frac{E_0^2}{E_d^2} + \frac{1}{2} Q_s b(1-\alpha)\, e^{-c}(2e^{-c}-1)\frac{E_0}{E_d} + \dots, \qquad (E_0 \ll E_d). \tag{4.29}$$

It may be of interest to note that for this case the relative variance may assume considerable values, as

$$\frac{\mathcal{D}^2(E_0)}{\mathcal{M}_1(E_0)} \sim \left(\frac{2}{3} - \frac{1}{2} Q_s b\right)(1-\alpha)\, e^{-c} \frac{E_0}{E_d} + 2e^{-c} - 1 + \dots, \qquad (E_0 \ll E_d). \tag{4.30}$$

Finally let us suppose that the energy spectrum of the bombarding neutrons will be congruent with that of fission neutrons, *i.e.* let

$$h(E_0) = \frac{1}{\sqrt{2\pi e E_f}} \exp\left[-\frac{E_0}{2E_f}\right] \sinh\sqrt{E_0/E_f}\,, \qquad (E_f = 0.5 \cdot 10^6 \text{ eV}). \tag{4.31}$$

On the basis of (3.8) the following expression will be obtained for the relative standard deviation of the number of lattice defects in a sample having infinite dimensions:

$$\frac{D(t)}{N_1(t)} \sim \sqrt{\left[0.625 + \frac{D_r^2(t)}{S_1(t)}\right]\frac{1}{S_1(t)}} \cdot \left\{1 + \left[0.195/\xi + \frac{0.027\, S_1(t) - 0.195\, D_r^2(t)/\xi}{0.625\, S_1(t) + D_r^2(t)}\right]\frac{E_d}{E_f} + \dots\right\}. \tag{4.32}$$

By considering the sample to be irradiated to be thin when compared against the mean free path of the bombarding neutrons in the expression (3.8) we have to calculate with $\mathcal{M}_1$ and $\mathcal{M}_2$ instead of M_1 and M_2. Let us suppose that Q_s is independent from E_0. In this case equations (4.27) and (4.28)

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will yield

$$(4.33)\qquad \frac{\mathcal{D}(t)}{\mathcal{N}_1(t)} \sim \sqrt{\left[\frac{2.167}{Q_s b} + \frac{D_r^2(t)}{S_1(t)} - 1\right]\frac{1}{S_1(t)}} \cdot \left\{1 + \frac{0.055}{(1-\alpha)[2.167 + Q_s b(D_r^2(t)/S_1(t) - 1)]}\frac{E_d}{E_f} + \ldots\right\}.$$

It is seen from expressions (4.32) and (4.33) that with increasing $S_1(t)$ the standard deviation of the number of displaced atoms decreases more slowly than that of Poisson's distribution.

It is clearly apparent from these expressions that the relative standard deviation in the number of displaced atoms is in fact higher than that computed on the basis of Poisson distribution, for usual doses of irradiation, however, it can be considered sufficiently low so that there is no need to worry about the statistically not significant character of the deviations in irradiation experiments carried out under similar physical conditions.

FEIN (5) has tried to reduce the divergence between theoretical estimations and experimental results. He assumed that the displacement limit energy E_d has no constant value but it will vary between given limits according to some statistical law.

Denote, according to FEIN, by $f(E_d)\,dE_d$ the probability of the displacement energy to assume values between E_d and $E_d + dE_d$. FEIN (5) has shown that in order to attain acceptable correspondence with experimental results a highly improbable large value E_d should be introduced for the maximum of the density function f.

In the opinion of the authors, in the interest of better correspondence between theory and experiment, the microphysical theory of the functions q and K entering our equations (2.6) and (2.7) should constitute the main objects of investigations, and comparisons with experimental results should be carried out with cautiousness.

RIASSUNTO (*)

Servendoci di metodi esatti della teoria della probabilità abbiamo determinato la funzione di distribuzione del numero degli spostamenti del reticolo nei solidi irradiati da particelle di alta energia. La funzione generatrice della funzione di distribuzione fornisce equazioni che determinano il numero medio degli spostamenti e la loro disper-

(*) *Traduzione a cura della Redazione.*

sione. Queste equazioni sono state risolte nell'ipotesi che le particelle primarie irradianti siano neutroni e che i fenomeni d'urto tra un atomo spostato e un atomo vibrante nel reticolo possano considerarsi come urti fra sfere rigide. Si è trovato che la variazione del numero degli spostamenti reticolari è maggiore della variazione della distribuzione di Poisson per il caso d'irradiazione con neutroni da fissione. Se lo spessore del campione irradiato è piccolo rispetto al cammino libero medio dei neutroni, la deviazione sistematica relativa è data da

$$\frac{\mathcal{D}(t)}{\mathcal{N}_1(t)} \sim \sqrt{\left[\frac{2.167}{Q_s b} + \frac{D_r^2(t)}{S_1(t)} - 1\right]\frac{1}{S_1(t)}}\left\{1 + \frac{0.055}{(1-\alpha)[2.167 + Q_s b\,(D_r^2(t)/S_1(t) - 1)]}\frac{E_d}{E_f} + \dots\right\},$$

dove $S_1(t)$ è il numero medio dei neutroni irradianti che penetrano nella superficie unitaria del campione in un tempo t e $D_r^2(t) = S_2(t) - S_1^2(t)$ la sua variazione. Q_s è la sezione d'urto macroscopica del materiale irradiato, b lo spessore del campione, E_d l'energia di spostamento in eV e $E_f = 0.5 \cdot 10^6$ eV. Finalmente $\alpha = ((M-1)/(M+1))^2$, dove M è il numero di massa del nucleo del campione. $\mathcal{N}_1(t)$ è il numero medio degli spostamenti reticolari nel tempo t.

Some Considerations on the Analysis of Primary Cosmic Ray Intensity Experiments (*).

A. Engler, M. F. Kaplon and J. Klarmann

Department of Physics and Astronomy, University of Rochester - Rochester, N.Y.

(ricevuto il 31 Dicembre 1958)

Summary. — A discussion is given of the procedures employed in the analysis of primary cosmic ray intensity experiments and the assumptions inherent in their use, with particular emphasis on nuclear emulsion as a detector. Explicit formulae for extrapolation of particle intensities for various geometries employed in emulsion work are given. Some discussion of geomagnetic effects is given and data are presented showing that the earth's shadow cone is not appreciably present at a geomagnetic latitude of 40°.

Introduction.

We present here some considerations concerning the deduction of primary cosmic ray particle intensities from high altitude experiments, with particular emphasis on nuclear emulsions as detectors. The problem is naturally separated into two distinct parts: the first (and our chief concern) is the deduction of primary particle intensities incident on the top of the earth's atmosphere from observations made at a finite depth; the second is the relationship of these intensities to those existing in our galaxy, far removed from our solar system. We feel that such a discussion serves a useful purpose since hitherto certain procedures have been used in analysis without any particular inquiry

(*) This research was supported in part by the United States Air Force through the Office of Scientific Research, Air Research and Development Command.

into their validity. Though it is still not possible to answer all the questions raised, it is important that the methods and the assumptions inherent in their use be clearly stated so that assessment and comparison of results can be done on a realistic basis.

1. – Basic concepts and procedures.

The basic particle intensity that is measurable in cosmic ray experiments seeking information on spectra is the unidirectional, differential particle intensity $J_k(h, \lambda, w; \theta, \varphi; E, t)$; its meaning is defined by the statement that $J_k\,\mathrm{d}S\,\mathrm{d}\Omega\,\mathrm{d}E\,\mathrm{d}t$ is the number of particles of type k, with kinetic energy per nucleon in the interval $\mathrm{d}E$ at E, incident in the time interval $\mathrm{d}t$ at t, from within the solid angle $\mathrm{d}\Omega(\theta, \varphi)$ upon an element of area $\mathrm{d}S$ placed normal to the direction of incidence at the point h, λ, w, where h is the depth from the top of the earth's atmosphere, and λ, w are the geomagnetic co-ordinates. Intimately related to the differential intensity J is the integral particle intensity

$$I_k(h, \lambda, w; \theta, \varphi; E, t) = \int\limits_E^\infty \mathrm{d}E'\, J_k(h, \lambda, w; \theta, \varphi; E', t)\,. \tag{1}$$

In general, in experiments to measure cosmic ray intensities, the detector has a finite area, different geometrical efficiencies for recording particles incident at the detector from different directions (as well as possible different efficiencies with respect to energy), the data are recorded usually over a finite time interval τ and the position (h, λ, w) may change over the time interval τ. The detector records either a differential (F_k) or integral (G_k) flux given by

$$F_k(\bar{h}, \bar{\lambda}, \bar{w}; \Omega_0; E, \bar{\tau}(T)) = \int\limits_T^{T+\tau} \mathrm{d}t \iint\limits_{S_0} \mathrm{d}S \iint\limits_{\Omega_0} \mathrm{d}\Omega \int\limits_E^{E+\Delta E} \mathrm{d}E'\, J_k(h(t), \lambda(t), w(t); \theta, \varphi; E', t)\cdot \xi_k(E') \cos\eta(\theta, \varphi)\,, \tag{2a}$$

$$G_k(\bar{h}, \bar{\lambda}, \bar{w}; \Omega_0; E, \bar{\tau}(T)) = \int\limits_T^{T+\tau} \mathrm{d}t \iint\limits_{S_0} \mathrm{d}S \iint\limits_{\Omega_0} \mathrm{d}\Omega \int\limits_E^{\infty} \mathrm{d}E'\, J_k(h(t), \lambda(t), w(t); \theta, \varphi; E', t)\cdot \xi_k(E') \cos\eta(\theta, \varphi)\,, \tag{2b}$$

where η is the angle between the normal to the element of area $\mathrm{d}S$ of the detector and the direction defining the solid angle $\mathrm{d}\Omega$, $\xi_k(E)$ is the efficiency for detecting particles of energy E of type k, Ω_0 indicates any geometrical constraints imposed on particle selection, S_0 is the area of the detector, T is the epoch time of the measurement (by this we mean time measured from some

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arbitrary origin on an astrophysical scale) and the barred quantities indicate a time average for the experimental conditions.

The quantities of primary interest are particle intensities or densities at the top of the earth's atmosphere and at distances far removed from any local disturbances. We denote particle densities by the symbol $M_k(\boldsymbol{r}, \boldsymbol{p}_k, T)$ and define them by the statement that $M_k(\boldsymbol{r}, \boldsymbol{p}_k, T)\,\mathrm{d}^3r\,\mathrm{d}^3p_k$ is the number of particles of type k in the volume element d^3r at $\boldsymbol{r}$ with momenta in d^3p_k at $\boldsymbol{p}_k$ at the time T. With our definition of differential particle intensities, the connecting relation is $M_k(\boldsymbol{r}, \boldsymbol{p}_k, T) = J_k(\boldsymbol{r}; \theta, \varphi,; E, T)/p_k^2$; the integral density

$$N_k(\boldsymbol{r}, E, T) = \int_E^\infty \mathrm{d}E' \iint \mathrm{d}\Omega \, \frac{J_k(\boldsymbol{r}; \theta, \varphi; E', T)}{v_k},$$

is also of interest. We will denote by the superscripts 0 and ∞ respectively quantities at the top of the earth's atmosphere ($h = 0$) and at distances far from the earth. It is clear then that we must first deduce the quantities $J_k^0(\lambda, w; \theta, \varphi; E, T)$ or $I_k^0(\lambda, w; \theta, \varphi; E, T)$ to proceed further; the quantities J_k^∞ or N_k^∞ classically are deduced from the J_k^0 or N_k^0 by applying Liouville's theorem (¹) assuming that the earth's static field alone determines particle trajectories.

We now turn our attention to the problem of obtaining intensities at the top of the atmosphere from flux measurements at a finite depth. In order to simplify the discussion we shall assume that the geomagnetic co-ordinates λ, w are essentially constant during the interval of measurement and that any time dependence of J_k or I_k during the interval τ is small compared to any variation with respect to T so that during the period τ of acceptance, the only quantity that may vary directly with time is h.

In principle the ideal experiment uses a detector with $\xi_k(E) = 1$, $\cos\eta = 1$, and Ω_0 and ΔE accepted sufficiently small so that with $h =$ constant, $F_k/(S_0\Omega_0\tau\,\Delta E) = J_k(h, \lambda, w; \theta, \varphi; E, T)$ or the equivalent statement for G_k and I_k. By then varying h one could obtain J_k or I_k as a function of h, holding all other parameters constant, and experimentally extrapolate in this way to J_k^0 or I_k^0. Such a type of experiment seems in principle attainable by application of counter techniques; examples of this approach are the work of WEBBER and McDONALD (²) and of LINSLEY (³); such experiments however suffer principally from the problem of studying the variation with h for small h (*i.e.* h must be small compared to the shortest characteristic length of the

(¹) G. LEMAITRE and M. S. VALLARTA: *Phys. Rev.*, **43**, 87 (1933); W. F. G. SWANN: *Phys. Rev.*, **44**, 224 (1933).

(²) W. R. WEBBER and F. B. McDONALD: *Phys. Rev.*, **100**, 1460 (1955).

(³) J. H. LINSLEY: *Phys. Rev.*, **101**, 829 (1956).

nuclear processes contributing to the modification of flux with depth), so that the experimental extrapolation is difficult and in practice to date, the extrapolation is carried out theoretically (we assume here the intrinsic resolution problem of the detector is solved and the apparatus has its orientation maintained). In principle, the problem could be approached in this fashion also by exposing oriented stacks of nuclear emulsion at various values of h; here $\cos\eta \neq 1$ for all directions, but its value is determinable and the analysis would proceed as above. By using a sufficiently large area, enough data could be obtained for this procedure if the variation with h could be experimentally obtained.

In practice these procedures are not followed and the principal methods used have been of two kinds. We shall discuss these methods here and the assumptions inherent in their use. Both methods involve an extrapolation to $h=0$. The first (hereafter called method A) is effectively an attempt at a procedure to substitute for the idealized one outlined above. In this method, the variation of F_k or G_k with zenith angle is studied; here, as θ is varied, the path length through the atmosphere $x=h/\cos\theta$ changes and the variation in x as a result of changing θ (assuming $h=$ constant) is viewed as an effective variation of h and the extrapolation is made in this way [4]. In the second method (method B) the extrapolation is made theoretically by taking into account the various processes which can contribute to the flux of particles of type k at a depth h in the atmosphere (essentially their diffusion is studied) [5,6]. These two procedures have inherent in their application certain assumptions concerning the nature of primary cosmic radiation, geomagnetic effects and nuclear interactions which seem not to be sufficiently stressed. We shall attempt here some discussion of these assumptions.

2. – The variation of flux with depth.

In order to make clear the nature of these assumptions, we first look at the physical processes involved in the passage of a beam of particles of type k through an absorber. Let us assume that the beam incident on the absorber is specified, so that initially we know $J_k^0(\lambda, w; \theta, \varphi; E, T)$; we are interested in determining $J_k(h, \lambda, w; \theta, \varphi; E, T) = J_k(h, \Omega, E)$ where the notation indicates the other parameters are assumed fixed. Thus we must ask what physical processes can contribute to $J_k(h, \Omega, E)\,\mathrm{d}h\,\mathrm{d}\Omega\,\mathrm{d}E$, the number of particles of type k in $\mathrm{d}h$ at h, within the solid angle $\mathrm{d}\Omega$ and having an energy in $\mathrm{d}E$ at E.

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(4) This procedure has been used in the work of DANIELSON *et al.*: *Phys. Rev.*, **103**, 1075 (1956) as well as by earlier workers.

(5) H. L. BRADT and B. PETERS: *Phys. Rev.*, **89**, 943 (1950).

(6) M. F. KAPLON, J. H. NOON and G. W. RACETTE: *Phys. Rev.*, **96**, 1408 (1954).

The following processes contribute: 1) A particle of type k initially in $\mathrm{d}\Omega$ and $\mathrm{d}E$ can scatter out of $\mathrm{d}\Omega$, or one in $\mathrm{d}E$ and $\mathrm{d}\Omega'$ can scatter into $\mathrm{d}\Omega$; 2) A particle initially in $\mathrm{d}\Omega\,\mathrm{d}E$ can be catastrophically absorbed; 3) A particle in $\mathrm{d}\Omega$ and $\mathrm{d}E'$ at E' can by energy loss or interaction enter the interval $\mathrm{d}E$ at E or one in $\mathrm{d}\Omega\,\mathrm{d}E$ can leave the interval by interaction or energy loss; 4) A particle of type $i \neq k$ in $\mathrm{d}\Omega'\mathrm{d}E'$ can by interaction produce a particle of type k in $\mathrm{d}\Omega\,\mathrm{d}E$ which enters $\mathrm{d}h$ at h. Assuming that the basic cross-sections are known, one could in principle write down the diffusion equation taking the above processes into account and seek a solution subject to given boundary conditions. In practice, certain of the above processes are ignored and simplifying assumptions made concerning the others. Usually process (1) is ignored which certainly seems justifiable for not too low momenta and in (4) the assumption is usually made that the direction is conserved in the interactions. The latter is certainly not true in general but the angular deviation from the original direction in interactions is approximately (7) $(4\langle T_k\rangle/(3A_k p_i))^{\frac{1}{2}}$, where T_k is the total kinetic energy of the produced particle k in the rest system of the interacting particle of momentum p_i per nucleon and A_k is the number of nucleons constituting particle k. Since $\langle T_k\rangle/(A_k p_i) \ll 1$ for $E_i \sim$ several hundred MeV/nucleon, we may ignore angular changes for reasonably high energies in (4). These assumptions lead to the consideration of a unidirectional diffusion equation. It is then simpler to consider the intensity $J_k(x, E)$, where x is the distance traveled in the absorber in the direction θ_0, $\varphi_0(h = x\cos\theta_0)$ and the boundary condition at $x = 0$ is $J_k^0(\lambda, w; \theta_0, \varphi_0; E, T)$. The diffusion equations are then:

$$(3a) \qquad \frac{\partial J_k(x, E)}{\partial x} = \sum_{i=k}^{z_{\max}} \int_E^{\infty} \mathrm{d}E'\, \Phi_{ik}(E', E)\, J_i(x, E') - J_k(x, E)\,\Phi_k(E) + \frac{\partial}{\partial E}\left(J_k(x, E)\,\varepsilon_k(E)\right),$$

where $\Phi_{ik}(E', E)$ is the probability per unit lenght that in an interaction of a particle of type i with energy E', a k-type particle of energy E is produced, $\Phi_k(E)$ is the probability per unit length that a particle of type k of energy E is catastrophically destroyed and $\varepsilon_k(E)$ is the assumed continuous energy loss per unit length of k-type particles of energy E. We specifically now imply an identification of particle by charge; *i.e.* $i = 1$ is a proton, etc., so that $\Phi_{ik} = 0$ for $i < k$ (we assume pickup processes are negligible compared with fragmentation). The first term on the right hand side accounts for process (4) plus the first part of (3) for $i = k$; the second term accounts for process (2).

(7) M. F. Kaplon *et al.*: *Phys. Rev.*, **85**, 295 (1952).

It is clear physically that no spatial stationary solutions $(J_k(x, E) = J_k(E)j(x))$ exist for this set of equations since the individual particles do not reproduce themselves. This means first of all that the depth dependence for a particular species k cannot be described by a simple attenuation for any reasonable form of $\Phi_{ik}(E', E)$ or $\Phi_k(E)$.

In order to proceed further we must have some additional information concerning the so-far unspecified parameters. Experiment and theory indicate that in the fragmentation of heavy nuclei into lighter components, the energy per nucleon is conserved to essentially the same order as direction. This suggests that it is reasonable to write $\Phi_{ik}(E', E) = \Phi_{ik}(E')\,\delta(E' - E)$; if we further write $\Phi_{ik}(E') = P_{i,k}(E')/\lambda_i(E')$ and $\Phi_k(E) = 1/\lambda_k(E)$, where λ_i represents an interaction mean free path for a particle of type i and $P_{i,k}$ is the average number of particles of type k produced in the interaction of a type i particle, we have

$$\text{(3b)}\qquad \frac{\partial J_k(x, E)}{\partial x} = \sum_{i=k}^{z_{\max}} P_{i,k}(E)\,J_i(x, E)/\lambda_i(E) - J_k(x, E)/\lambda_k(E) + \frac{\partial}{\partial E}\left(J_k(x, E)\,\varepsilon_k(E)\right) .$$

This is essentially the form of the diffusion equations as they have appeared previously [6], except for the implicit energy dependence of the fragmentation probabilities $(P_{i,k})$ and the interaction mean free paths λ_i and the inclusion of an ionization loss term. The non-existence of stationary solutions (even neglecting ionization loss and energy dependence) is manifestly evident here. The only simple solution which exists for $J_k(x, E)$ is that obtained by neglecting ionization loss and assuming the $P_{i,k}$ and λ_i independent of energy. For this case a solution stationary in energy $J_k(x, E) = j(E)S_k(x)$ can be obtained

$$\text{(4)}\qquad S_k(x) = S_k^0 \exp[-x/\lambda'_k] + \sum_{i=k+1}^{z_{\max}} A_{ik}(S_i^0 \exp[-x/\lambda'_k] - S_i(x)) ,$$

where

$$\lambda'_i = \lambda_i/(1 - P_{i,i}) , \qquad \alpha_{ij} = \lambda'_i\lambda'_j/(\lambda'_j - \lambda'_i) > 0$$

and

$$A_{ik} = (\alpha_{ik}/\lambda_i)\Big(P_{i,k} + \sum_{j=k+1}^{i-1} P_{i,j}P_{j,k}\alpha_{jk}/\lambda_j + \dots + P_{i,i-1}\dots P_{k+1,k}\alpha_{i-1,k}\dots\alpha_{k+1,k}/(\lambda_{i-1}\dots\lambda_{k+1})\Big) ,$$

with the boundary condition $j(E)S_k^0 = J_k^c(\lambda, w; \theta_0, \varphi_0; E, T)$.

For the case of the integral spectrum, the equation neglecting energy dependence of the interaction parameters is

$$\text{(5)}\qquad \frac{\partial I_k(x, E)}{\partial x} = \sum_{i=k}^{z_{\max}} \frac{P_{i,k}I_i(x, E)}{\lambda_i} - \frac{I_k(x, E)}{\lambda_k} - J_k(x, E)\,\varepsilon_k(E) .$$

In first approximation, the last term is again neglected and a solution $I_k(x, E) = i(E)S_k(x)$ is assumed; $S_k(x)$ is identical so that of (4). The boundary condition is for this case $i(E)S_k^0 = I_k^0(\lambda, w; \theta_0, \varphi_0; E, T)$.

For our discussion the solution for the stationary energy case suffices. Expressed in terms of intensities we have

$$I_k(h, \lambda, w; \theta, \varphi; E, T) = I_k^0(\lambda, w; \theta, \varphi; E, T) \exp[-h/(\lambda_k' \cos\theta)] + \tag{6}$$

$$+ \sum_{i=k+1}^{k_{\max}} A_{ik}\left(I_i^0(\lambda, w; \theta, \varphi; E, T) \exp[-h/(\lambda_k' \cos\theta)] - I_i(h, \lambda, w; \theta, \varphi; E, T)\right),$$

with a similar expression for J_k. We emphasize that this solution has inherent in it the assumption that for the fixed direction θ, φ at λ, w, all components i have the same energy spectrum.

Experimentally it has not been possible so far to realize practically a basic assumption implicit in (6): this is the delineation of the charge spectrum in the detail implied. Practically speaking the proton ($i = 1$) and He ($i = 2$) components are well defined observationally but for reasons of insufficient intensity (essentially for all components with $Z > 2$) it has been customary to lump the nuclei into charge groups: *e.g.* L nuclei ($3 \leqslant i \leqslant 5$); M nuclei ($6 \leqslant i \leqslant 9$); H nuclei ($i \geqslant 10$), the latter being sometimes further subdivided into two groups. The diffusion equations governing the propagation of charge groups have been written [6] (we use capital letters to designate charge groups) as

$$\frac{\mathrm{d}I_J(x)}{\mathrm{d}x} = \sum_{I \geq J} \frac{P_{I,J}}{\lambda_I} I_I(x) - \frac{I_J(x)}{\lambda_J}\,, \tag{5b}$$

which has a solution like (6) with an appropriate change of notation. In addition to all the assumptions inherent so far this has the additional one that each group is presumed well characterized by its assigned values of λ_I and $\lambda_{I,J}$, which are assumed to be constant.

Now the conditions for compatibility of (5b) and (5) (without energy loss) define the parameters $P_{I,J}$ and λ_I as follows:

$$1/\lambda_I = \Big(\sum_{i=I_{\min}}^{I_{\max}} I_i(x)/\lambda_i\Big)\Big/\Big(\sum_{i=I_{\min}}^{I_{\max}} I_i(x)\Big)\,,$$

$$P_{I,J} = \Big(\sum_{i=I_{\min}}^{I_{\max}} P_{i,J} I_i(x)/\lambda_i\Big)\Big/\Big(\sum_{i=I_{\min}}^{I_{\max}} I_i(x)/\lambda_i\Big)\;; \qquad P_{i,J} = \sum_{k=J_{\min}}^{J_{\max}} P_{i,k}\,,$$

$$1/\lambda_I' = (1 - P_{I,I})/\lambda_I = \Big(\sum_{i=I_{\min}}^{I_{\max}} (1 - P_{i,I}) I_i(x)/\lambda_i\Big)\Big/\Big(\sum_{i=I_{\min}}^{I_{\max}} I_i(x)\Big)\;; \qquad P_{i,I} = \sum_{k=I_{\min}}^{i} P_{i,k}\,.$$

The above imply that the group values λ_I, $P_{I,J}$, λ_I' are depth dependent (since the density functions are depth dependent) and are sensitive to the consti-

tution of the group. If the experimental determination were made on a sample that had traversed a fixed depth this would (as the computation is normally made) approximately correspond to the weighting above at that depth but in practice particles are accepted over a wide range of x (various zenith angles) and this does not correspond to the definition above. We obtain an estimate of the sensitivity to constitution of the parameters involved as follows.

Using the data of Bristol ([8]) and Turin ([9]) groups for individual fragmentation probabilities in emulsion, we have calculated, using the above relations, λ'_L, λ_M, λ'_M, $P_{M,L}$, λ_H, λ'_H, $P_{H,L}$, λ'_S (S means $Z \geq 6$) and the constant $K = (\sum_{i\geq 6} P_{i,L} I_i/\lambda_i)/(\sum_{i\geq 6} I_i)$ for various constitutions of the groups L, M, H and S for all emulsion interactions. The purpose of this is to obtain an idea of the variation of these parameters with the assumed composition of the group in order to see how valid is the assumption that they are constant. λ'_L is most sensitive to the amount of boron in this group since its fragmentation will contribute principally to its attenuation. Varying the boron component by a factor of two changed λ'_L from 64.6 to 65 g/cm² (of emulsion). For the medium group large variations in relative abundance within the group reflected very little change ($< 2\%$) in λ_M and λ'_M from that experimentally observed while $P_{M,L}$ varied at the same time between 0.154 (for the largest carbon abundance) and 0.181 (the smallest carbon abundance), Similarly in the H group λ_H and λ'_H were constant to within 6% over large variations in composition while $P_{H,L}$ was constant to within 5%.

For the parameters λ'_S and K the specific variation used corresponded to a weighting within this group proportional to $(\lambda_i/\lambda_c)^n\cdot$(observed composition in emulsion), where values for n of 0, 1, 2, and 4 were used. We found for these n values $\lambda'_S = 62$, 61.1, 60.2 and 58.1 g/cm² (of emulsion) respectively; the n values correspond to a ratio N_H/N_M of 0.49, 0.36, 0.27 and 0.16 respectively. The parameter 100 K for these n values was 0.33, 0.34, 0.33 and 0.31 respectively. It would appear from the relatively slight sensitivity of λ'_S to composition that the assumption that the S group is simply attenuated is valid. If however the diffusion equations are written as in the analysis by the Bombay group ([10])

$$\frac{\mathrm{d}I_S(x)}{\mathrm{d}x} = -I_S(x)/\lambda'_S; \qquad \frac{\mathrm{d}I_L(x)}{\mathrm{d}x} = -I_L(x)/\lambda'_L + KI_S(x)\,,$$

([8]) C. J. Waddington: *Phil. Mag.*, **21**, 1059 (1957); V. Y. Rajopadhye and C. J. Waddington: *Phil. Mag.*, **25**, 19 (1958).

([9]) R. Cester, A. Debenedetti, C. M. Garelli, B. Quassiati, L. Tallone and M. Vigone: *Nuovo Cimento*, **7**, 371 (1958).

([10]) M. V. K. Appa Rao *et al.*: to appear in *Phys. Rev.* We are indebted to Prof. Peters for a preprint of this paper.

the parameter $1/\lambda = 1/\lambda'_S - 1/\lambda'_L$ appears in the solution and is assumed constant. It takes on the values 133.5, 105, 83.5 and 57.2 g/cm² (of emulsion) respectively for $n = 0, 1, 2$ and 4. Though a similar parameter $1/\alpha_{IJ} = 1/\lambda'_I - 1/\lambda'_J$ appears in the formulation of three groups, it is much less sensitive to the variations of the λ'_I with composition owing to the wide differences between them. If a similar type behavior of the various parameters holds for air (for which the data at present are insufficient to warrant a detailed analysis) one may conclude that the three group division is a reasonable approximation but that the two group division may be quite sensitive in analysis to the composition as a function of depth.

For the continuing discussion we will not further explicitly distinguish between the three (or more) group separation of the heavy nuclei or the individual characterization but will assume that the parameters are well defined in any case.

3. – Applications.

In the employment of method A, the most stringent application would be that the apparatus is oriented in azimuth (φ) and that the zenith angle (θ) is varied for fixed azimuth. Implicit in this is the assumption that the stationary energy dependence is independent of zenith angle, *i.e.* that for each component the same energy spectrum is incident at $h = 0$ for every zenith angle with the same intensity. If the detector is not oriented in azimuth, this implies the more stringent assumption of isotropy at incidence. As mentioned previously there is an additional difficulty in extrapolating to $h = 0$ in this way in that the intensity is not simply attenuated.

Let us consider method B. If the apparatus is oriented and the solid angle of acceptance sufficiently small, a direct extrapolation back can be made for each direction (θ, φ). This would introduce no additional assumptions than those involved in the diffusion equations. If no azimuth orientation is maintained, this requires the assumption in general that J_k^0 or I_k^0 are azimuth independent. If, as is usually done, data are accepted over a finite solid angle, this implies the assumption at $h = 0$ of isotropy over this range (the validity of these assumptions will be discussed later). In the above discussion of methods A and B we have assumed energy independence of interactions; if this is not valid, the procedures are even more susceptible to error. It should be pointed out that any extrapolation procedure of the B type that determines an intensity at altitude h, accepting particles over a finite solid angle, and extrapolates this intensity back to $h = 0$ unidirectionally, assuming some mean zenith angle of incidence, is incorrect. The correct procedure is

to form

$$G_k(\bar{h}, \bar{\lambda}, \bar{w}; \Omega_0, E, \bar{\tau}(T)) = \tag{7}$$

$$= \int_T^{T+\tau} \mathrm{d}t \iint_{S_0} \mathrm{d}S \iint_{\Omega_0} \mathrm{d}\Omega \, \{ I_k^0(\lambda, w; \theta, \varphi; E, t) \exp[-h/(\lambda_k' \cos\theta)] -$$

$$+ \sum_{i=k+1}^{Z_{\max}} A_{ik} I_i^0(\lambda, w; \theta, \varphi; E, t) \exp[-h/(\lambda_k' \cos\theta)] \} \cos\eta(\theta, \varphi) -$$

$$- \sum_{i=k+1}^{Z_{\max}} A_{ik} G_i(h, \lambda, w; \Omega_0; E, \bar{\tau}(T)) \,,$$

or a similar equation for F_k in terms of J_k^0. Here G_i and G_k are measured quantities and the integrals can be evaluated assuming either isotropy or some given angular dependence at the top of the atmosphere: there results then a system of simultaneous algebraic equations for the determination of I_i^0 or (J_i^0). The assumption [8,9,11] that

$$(1/\Omega_0) \iint_{\Omega_0} \mathrm{d}\Omega \exp[-h/(\lambda_I \cos\theta)] \cos\eta(\theta, \varphi) = \exp[-h/(\lambda_I \langle\cos\theta\rangle_{\Omega_0})] \langle\cos\eta\rangle_{\Omega_0} \,,$$

is incorrect in principle even assuming isotropy.

4. – Geometrical considerations.

It is possible to give more specific formulae for the case of nuclear emulsion as a detector since there are essentially three distinct geometries employed: 1) Horizontal, *i.e.* normal to emulsion plane in the zenith direction so that $\eta = \theta$; 2) Vertical geometry utilizing a line scan so that again $\eta = \theta$; 3) Vertical geometry utilizing an area scan so that if the emulsion normal is taken in say the x direction, $\cos\eta = \pm \sin\theta \cos\varphi$. In general there are constraints imposed with respect to minimum track length accepted. If L_0 is the minimum acceptable projected track length in emulsion and D the unprocessed emulsion thickness, the constraints may be expressed as

1) φ unrestricted; $\operatorname{tg}\theta \geqslant \cot\alpha_0$;

2) and 3) φ unrestricted for $\theta \leqslant \alpha_0$

$$|\cos\varphi| \leqslant \sin\alpha_0 / \sin\theta \qquad \text{for} \qquad \theta > \alpha_0; \quad \operatorname{tg}\alpha_0 = D/L_0$$

([11]) C. J. Waddington: *Nuovo Cimento*, **3**, 930 (1956).

In addition there may be a constraint on zenith angle acceptance so that only those $\theta \leqslant \theta_1$ are accepted. The above conditions then define Ω_0.

It is also possible to take into account any variation in h during the exposure time by approximating the flight curve by a series of straight line segments; for each segment covering a time interval $\Delta\tau_i$ we have $h(t) = h_i + (h_{i+1} - h_i)t/\Delta\tau_i$ and $\sum_i \Delta\tau_i = \tau$. The effect is to replace $\int_T^{T+\tau} \mathrm{d}t$ with $\sum_i \int_0^{\Delta\tau_i} \mathrm{d}t$ and h everywhere by $h(t)$ defined above. It is then most convenient to do the time integration first and then the integration on solid angle.

Equation (7) can then be represented as follows, where for convenience we have suppressed $\bar{h}$, $\bar{\lambda}$, $\bar{w}$, Ω_0 and $\bar{\tau}$ and represented $I^0(h, \lambda, w; \theta, \varphi; E, T)$ by $I^0(E)$ assumed isotropic in the upper hemisphere; the summation on the index σ is over the level portions of the flight. The index γ refers for $\gamma = 1, 2$ and 3 respectively to case (1), (2) and (3) above,

$$(8)\quad G_k(E) + \sum_{i=k+1}^{Z_{\max}} A_{ik} G_i(E) = C_\gamma S_0 \Big\{ \sum_\sigma \Delta\tau_\sigma H^1_\gamma(h_\sigma/\lambda'_k, \cos\theta_1, \beta_\gamma)\big[I^0_k(E) + \sum_{i=k+1}^{Z_{\max}} A_{ik} I^0_i(E)\big] + \\ + \sum_s \sum_{j=0}^{1} (-)^j \lambda'_k \Delta\tau_s/(h_{s+1} - h_s) H^2_\gamma(h_{s+j}/\lambda'_k, \cos\theta_1, \beta_\gamma)\big[I^0_k(E) + \sum_{i=k+1}^{Z_{\max}} A_{ik} I^0_i(E)\big]\Big\},$$

where $C_\gamma = 2\pi$ for $\gamma = 1, 2$ and $C_3 = 4$; $\beta_1 = \sin\alpha_0$, $\beta_2 = \cos\alpha_0$ and $\beta_3 = 0$. The functions $H^i_\gamma(x, y, z)$ are defined below as

$$(9a)\qquad H^1_1(x, y, z) = (1/2)\big\{z^2 \exp[-x/z](1 - x/z) - y^2 \exp[-x/y](1 - x/y) + \\ + x^2\big(Ei(-x/y) - Ei(-x/z)\big)\big\},$$

$$(9b)\qquad H^2_1(x, y, z) = (1/3)\big\{z^3 \exp[-x/z] - y^3 \exp[-x/y] - xH^1_1(x, y, z)\big\},$$

$$(9c)\qquad H^1_2(x, y, z) = H^1_1(x, y\cos\alpha_0/\cos\theta_1, z/\cos\alpha_0) + \\ + (2/\pi)\int_y^z s\exp[-x/s]\sin^{-1}\left(\frac{1-z^2}{1-s^2}\right)^{\frac{1}{2}} \mathrm{d}s,$$

$$(9d)\qquad H^2_2(x, y, z) = H^2_1(x, y\cos\alpha_0/\cos\theta_1, z/\cos\alpha_0) + \\ + (2/\pi)\int_y^z s^2\exp[-x/s]\sin^{-1}\left(\frac{1-z^2}{1-s^2}\right)^{\frac{1}{2}} \mathrm{d}s;$$

$$(9e)\qquad H^1_3(x, y, 0) = \\ = \int_y^1 \mathrm{d}s(1-s^2)^{\frac{1}{2}}\exp[-x/s] - \cos^3\alpha_0 \int_{y/\cos\alpha_0}^1 \mathrm{d}s(1-s^2)^{\frac{1}{2}}\exp[-x/(s\cos\alpha_0)],$$

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(9f) $H_3^2(x, y, 0) =$

$$= \int_y^1 s\, ds (1-s^2)^{\frac{1}{2}} \exp[-x/s] - \cos^3\alpha_0 \int_{y/\cos\alpha_0}^1 s\, ds(1-s^2)^{\frac{1}{2}} \exp[-x/(s\cos\alpha_0)]\,.$$

It is worth commenting that the expression for $G_k(E)$ also defines the integral (projected) track length distribution due to the constraint imposed that $L_p > L_0$. This has been stated as being proportional, for an area scan, to $1/L_0^2$ and for a line scan to $1/L_0$ under the assumption of an isotropic intensity at the detector (emulsion) and in practice the agreement of experimental integral track length distributions with this has been used as a test of scanning efficiency. In fact, the statement is only approximately true, even for an isotropic intensity at the detector, a condition not satisfied in practice when the detector is at a finite depth in the atmosphere. Assuming an isotropic intensity (I_0) at the detector, there follows from (9-a, c, e) with $x=0$ and $y=\cos\theta_1$, the integral track length distributions $N_\gamma(L_p \geqslant L_0)$

(10a) $N_1 = (\pi I_0 D^2)/(L_0^2 + D^2)(\sin^2\theta_0 - (L_0/D^2)\cos^2\theta_1)\,;\qquad \operatorname{tg}\theta_1 > (L_0/D)\,,$

$= 0$ otherwise,

(10b) $N_2 = 2I_0\{\sin^2\theta_1 \sin^{-1}(D/(\sin\theta_1(L_0^2+D^2)^{\frac{1}{2}})) +$

$D[(\sin^2\theta_1 - (D^2/(L_0^2+D^2))^{\frac{1}{2}}\cdot(1/(L_0^2+D^2)^{\frac{1}{2}})]\}\,,$

(10c) $N_3 = ((\pi I_0 D^2)/(L_0^2+D^2))\{1 + (2(L_0^2+D^2)/(\pi D^2))\cdot$

$\cdot[(\cos\theta_1((\sin^2\theta_1 - D^2)/(L_0^2+D^2))^{\frac{1}{2}} - \sin\theta_1) +$

$+ L_0^2/(L_0^2+D^2)\sin^{-1}(\cos\theta_1(D^2+L_0^2)^{\frac{1}{2}}/L_0) - \sin^{-1}(\cos\theta_1)]\}\,,$

for $\operatorname{tg}\theta_1 > (D/L_0)$ and 0 otherwise for cases 2 and 3.

In cases (1) and (3) for $\theta_1 = \pi/2$ we have $N_{1,2} = \pi I_0 D^2/(L_0^2 + D^2)$ which is approximately $1/L_0^2$ for $(D/L_0)^2 \ll 1$; for case (2) with $\theta_1 = \pi/2$ and $(D/L_0)^2 \ll 1$ we have the distribution proportional to $1/L_0$, while for $D/(\sin\theta_1(L_0^2+D^2)^{\frac{1}{2}}) \ll 1$ we find $N_3 \approx 4I_0 \sin\theta_1(D/L_0)$. For the physical situation at a finite depth in the atmosphere h, the assumption of isotropy at the detector is not valid and deviations from the limiting distributions above should occur. The physically realized distribution for a particle of type k is given by (7). This is seen to be dependent on the fluxes of particles of type $i > k$ which can give k-type particles through interactions and on the explicit definition of Ω_0 employed in the experiment.

5. – Geomagnetic effects.

We now consider the principal assumption that the radiation is isotropic and has a stationary energy spectrum at the top of the atmosphere. Assume that at infinity it is isotropic with a stationary energy spectrum and that the only factor influencing its approach to the earth is the earth's dipole magnetic field. For simplicity consider that in first approximation this effect is describable in terms of the Störmer cone. Then at a latitude λ, zenith angle θ and azimuth φ only those particles with momenta (per nucleon) greater than $p_0(\lambda, \theta, \varphi)$ may arrive [12] where

$$cp_0 = 4cp_v(\lambda)[1 + (1 - \sin\theta \cos\varphi \cos^3\lambda)^{\frac{1}{2}}]^{-2}, \tag{11a}$$

where $cp_v(\lambda) = 63.4Z \cos^4\lambda/(4A)$ is the cut-off in the vertical direction expressed in units of the proton rest energy. This can be approximated (for $\lambda \geqslant 30°$) by

$$cp_0 \approx cp_v(\lambda)[1 + \sin\theta \cos\varphi \cos^3\lambda/4]^2 . \tag{11b}$$

If we now assume an integral intensity at infinity

$$I_k = C_k/(1 + E_k)^{1+\gamma}$$

then due to the influence of the earth's field this will appear as

$$\left\{\begin{aligned} I_k^0(\lambda, w; \theta, \varphi; E, T) &= C_k/(1+E)^{1+\gamma} \qquad &&\text{for} \quad 1 + E \geqslant [1 + (cp_0)^2]^{\frac{1}{2}} \\ &= 0 &&\text{for} \quad 1 + E < [1 + (cp_0)^2]^{\frac{1}{2}} . \end{aligned}\right. \tag{12}$$

A similar statement holds for J_k^0. It is clear then that the assumption of a stationary (in energy) solution, isotropic at $h = 0$ is not valid.

Consider next the case in which the detector presents the same effective area at a fixed θ for the azimuths φ and $\varphi + \pi$, *i.e.* $\cos\eta(\theta, \varphi) \cos\eta = (\theta, \varphi + \pi)$; this is the case for nuclear emulsions in vertical or horizontal orientation. We then have

$$\begin{aligned} \cos\eta\,[I_k^0(\theta, \varphi) + I_k^0(\theta, \varphi + \pi)] &= C_k \cos\eta \left[\frac{1}{(1 + [cp_0(\theta, \varphi)]^2)^{(1+\gamma)/2}} + \right. \\ &\left. + \frac{1}{(1 + [cp_0(\theta, p + \pi)]^2)^{(1+\gamma)/2}}\right] \approx 2C_k \cos\eta/(1 + [cp_v(\lambda)]^2)^{(1+\gamma)/2} \cdot \\ &\cdot \left\{\frac{1}{[1 - (cp_v(\lambda) \sin\theta \cos\varphi \cos^3\lambda)/(1 + [cp_v(\lambda)]^2)^2]^{(1+\gamma)/2}}\right\} . \end{aligned} \tag{13}$$

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[12] R. A. ALPHER: *Journ. Geophys. Res.*, **55**, 437 (1950).

To a first approximation, on the above assumptions, the effective intensity accepted without bias from φ and $\varphi+\pi$ at a fixed θ is independent of both θ and φ and the form of the assumed spectra is conserved, corresponding to an effective cut-off energy $E_0 = [1+(cp_v(\lambda))^2]^{\frac{1}{2}} - 1$. It is worth emphasizing that only in this approximation is the form of the spectra conserved. It is clear that a direct measurement of the differential spectrum can yield misleading results interpreted in terms of $J_k^0(E)$ if an azimuthal average is performed, since it will yield less low energy particles than would be present in the absence of the earth's field unless this effect is explicitly folded in. The usual statement (4) concerning the independence of the integral spectrum averaged over azimuth follows from the above; as demonstrated in ref. (4), it is valid for $\lambda \leqslant 30°$.

If, however, one examines the predictions of geomagnetic theory, *e.g.* as given by ALPHER (12), it appears that the simple Störmer cone is not valid and is quite distorted by the earth's shadow cone; this distortion increases as λ increases and for a fixed λ with increasing zenith angle. The influence of more rigorous details of the theory are more difficult to assess due to the lack of complete orbit calculations, so we restrict ourselves to the effect of the shadow cone, assuming a centered dipole. This effect is presented in Fig. 1 for a latitude of 40°; here we have taken (for nuclei of $A=2Z$) the predictions of the cut-off presented by the Störmer plus earth's shadow cone as given by ALPHER and averaged a spectrum of the form $1/(1+E)^{1.6}$ over azimuth for a fixed zenith angle in intervals of 10° for zenith angles. For comparison we portray in the same figure a similar computation using the Störmer cone only.

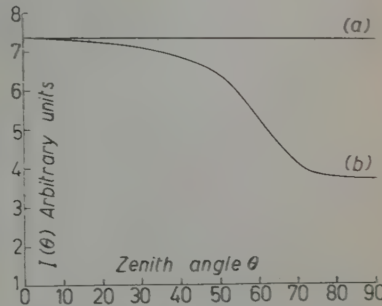

Fig. 1. – Integral intensity (averaged over azimuth) as a function of zenith angle, for geomagnetic latitude 40°, assuming (*a*) Störmer cone only and (*b*) Störmer plus earth's shadow cone as presented by ALPHER. A power spectrum of the form $1/(1+E)^{1.6}$ was assumed.

It seems clear that the assumption of a stationary integral energy spectrum (averaged over azimuth) is not acceptable if the earth's shadow cone is taken into account according to the results of the theory as presented by ALPHER (12).

It is however possible to test experimentally whether the predictions based on the shadow cone are correct. Consider an experiment, *e.g.* nuclear emulsions, in which the geometrical condition on η is satisfied and data are accepted without bias from all azimuths. In the determination of the flux at the top of the atmosphere we can place the additional constraint on Ω_0 that

only tracks with zenith angle $\theta \leqslant \theta'$ are accepted; we denote the intensity so determined by $\langle I^0 \rangle_{\theta'}$. We can then form

$$\mu(\theta', \theta'') = \frac{\langle I^0 \rangle_{\theta'} - \langle I^0 \rangle_{\theta''}}{\langle I^0 \rangle_{\theta'} + \langle I^0 \rangle_{\theta''}}, \tag{14}$$

and compare the experimental value of μ with that calculated according to the predictions of geomagnetic theory, assuming a reasonable integral energy spectrum (we used that quoted above). For $\lambda = 40°$ we find $\mu^{\text{th}}\,(\pi/4, \pi/2) = 0.13$ for the Störmer plus shadow cone and essentially 0 for the Störmer cone alone. Experimentally (13) we find $\mu^{\text{exp}}(\pi/4, \pi/2) = 0.02 \pm 0.015$ where for the latter determination we have used all particles of $Z \geqslant 2$ in order to obtain increased statistical weight. If μ^{exp} is determined for α-particles and the individual charge groups a similar result obtains but with decreased statistical accuracy. This result strongly suggests that the earth's shadow cone predictions for the aligned dipole are incorrect and seem more in accord with the simple Störmer cone (14,16). It appears then that the integral spectrum, azimuthally averaged at a latitude of 40° is stationary with respect to energy for energies $\geqslant$ 1.2 GeV/nucleon for nuclei of $A = 2Z$.

The experimental observation above, as well as the others of WINCKLER (14) and DANIELSON (15), would appear to justify for integral spectra determinations, the acceptance of particles from all azimuths if the conditions given previously are satisfied. However we should like to emphasize again that these results do not justify the same procedures for the determination of differential energy spectra and we believe that such procedures will tend to bias against lower energy particles. In the determination of differential spectra at high latitudes where evidence (17,18) has been presented for the existence of particles below the prima-facie geomagnetic cut-off, it would seem at the moment that the only well defined procedure would be to accept particles in a fairly well defined azimuthal region, since the nature of the perturbing forces is unknown. One might hope that if careful enough measurements were made some information concerning the properties of this perturbing field could be deduced from the azimuthal dependence of the differential energy spectra, if any exists.

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(13) A. ENGLER *et al.*: to appear in *Phys. Rev.*
(14) J. R. WINCKLER and K. A. ANDERSON: *Phys. Rev.*, **93**, 596 (1954).
(15) R. E. DANIELSON and P. S. FREIER: *Phys. Rev.*, **109**, 151 (1958).
(16) M. SCHWARTZ: private communication and *Bull. Am. Phys. Soc.*, **1**, 319 (1956).
(17) P. H. FOWLER *et al.*: *Phil. Mag.*, **2**, 157 (1958).
(18) F. B. McDONALD: *Phys. Rev.*, **107**, 1386 (1957).

6. – Cosmic rays at distances far from the solar system.

Of principal interest for cosmological considerations are the particle intensities or densities at distances well removed from our local part of the galaxy. As mentioned previously these have been deduced by an application of Liouville's theorem. The particle density M_k is the appropriate quantity for discussion here since it is for this that $\mathrm{d}M_k/\mathrm{d}t = 0$ along trajectories in phase space. As originally pointed out by SWANN (1) the number of systems in a volume element of phase space is $M_k\,\mathrm{d}^3r\,\mathrm{d}^3\pi$, where $\boldsymbol{\pi}$ is the canonical momentum but the Jacobian $J(\boldsymbol{\pi}, \boldsymbol{r}/\boldsymbol{p}, \boldsymbol{r}) = 1$ so that the quantity $M_k\,\mathrm{d}^3r\,\mathrm{d}^3p$ is also conserved; this result is valid even for time dependent fields. Its usefulness follows only if the dynamically allowed trajectories are known: the well known case of the motion of a charged particle in the field of a static magnetic dipole is sufficiently tractable for the theorem to be useful (1). For this case $|\boldsymbol{p}|$ is a constant of the motion and thus $J_k\,\mathrm{d}^3r\,\mathrm{d}^3p$ is conserved: thus the intensity along the trajectories is constant and $J_k^0 = J_k^\infty$. From such arguments it has been deduced that the cosmic ray densities at very large distances are homogeneous and isotropic. In light of the post-war advances in our knowledge of the properties of cosmic radiation and of various factors influencing them in their propagation one can well question the validity of the usual application of Liouville's theorem and the inference concerning J_k.

As a specific example we can consider the situation if a Fermi-type (19) acceleration mechanism should exist in the galaxy. Here the particles are being continuously accelerated by time varying magnetic fields and the dynamical description is necessarily statistical: thus well defined trajectories are not attainable since we have only a statistical description of the fields in the galaxy. It is true that the model in itself implies homogeneity and isotropy (to the extent that loss at the boundaries can be neglected) but it is not clear that this can be inferred from an application of Liouville's theorem. At a different extreme we can consider the inferences of the observations of the change in energy spectrum observed over the solar cycle (20). PARKER (21) presents an interesting interpretation of this in terms of the diffusion of an external (to the solar system) cosmic ray intensity through a plasma connected to the solar system and intimately correlated with solar activity. If we restrict ourselves to cosmic rays of energy less than ~ 30 GeV (in order to connect with reality) such a model could predict the observation of an isotropic flux incident on the earth even if the external flux were unidirectional.

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(19) E. FERMI: *Phys. Rev.*, **75**, 1169 (1949).
(20) H. V. NEHER and H. ANDERSON: *Phys. Rev.*, **109**, 608 (1958).
(21) E. N. PARKER: private communication.

In fact the observations at non-impact zones during the great flare of 23 Feb., 1956 [22] give direct experimental support for this view. If one would imagine that the flare repeats itself at sufficiently close intervals of time, the non-impact zone observations would correspond to the situation above.

Our point here is to emphasize that it may be somewhat questionable to infer the galactic spatial and angular distribution of cosmic rays from measurements within our solar system (this statement may quite easily be an energy-sensitive one, its validity being inversely proportional to the energy under consideration). We feel that the statements concerning the isotropy and distribution in space of cosmic radiation, reasonable as they seem, represent (particularly for lower rigidity particles) implications of a specific model and do not appear demonstrably derivable from experiment at this time.

([22]) P. Meyer, E. N. Parker and J. A. Simpson: *Phys. Rev.*, **104**, 768 (1956).

RIASSUNTO (*)

Discussione dei procedimenti impiegati nell'analisi degli esperimenti sull'intensità dei raggi cosmici primari e le ipotesi inerenti al loro impiego con speciale riferimento alle emulsioni nucleari come rivelatori. Si danno formole esplicite per l'estrapolazione dell'intensità delle particelle per le diverse geometrie adottate lavorando con emulsioni. Si discutono alcuni effetti geomagnetici e si presentano dati che dimostrano che il cono d'ombra terrestre non interviene apprezzabilmente alla latitudine geomagnetica di 40°.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XII, N. 3 1° Maggio 1959

Muon Capture in Certain Light Nuclei (*).

A. Fujii

Department of Physics, Purdue University - Lafayette, Ind.

H. Primakoff

Department of Physics, Washington University - St. Louis, Mo.

(ricevuto l'11 Gennaio 1959)

Summary. — The rates of the muon capture reactions: $\mu^- + {}^3_2\text{He} \to {}^3_1\text{H} + \nu$, $\mu^- + {}^6_3\text{Li} \to {}^6_2\text{He} + \nu$, $\mu^- + {}^{12}_6\text{C} \to {}^{12}_5\text{B} + \nu$, with the daughter nuclei formed in their ground states, are calculated assuming « universality » between muon-bare nucleon and electron-bare nucleon coupling constants. The « induced » pseudoscalar interaction and the additional terms arising from the assumption of a « conserved vector current » are included in the muon-nucleon effective Hamiltonian. The ratios of the nuclear matrix elements for the muon captures and for the corresponding β-decays are first estimated in an approximately « model independent » fashion using appropriate nuclear proton density distribution functions and then evaluated in more detail on the basis of variational trial wave functions for ${}^3_2\text{He}$, ${}^3_1\text{H}$, and of LS coupling and jj coupling shell model wave functions (with configuration mixing) for ${}^6_3\text{Li}$, ${}^6_2\text{He}$, and ${}^{12}_6\text{C}$, ${}^{12}_5\text{B}$, respectively. The calculated capture rate for $\mu^- + {}^{12}_6\text{C} \to {}^{12}_5\text{B} + \nu$ is in agreement with experiment — experiments however are not yet available in the other two cases.

1. – Introduction.

The transition rate for nuclear capture of orbitally bound negative muons has been studied both theoretically and experimentally by several authors (1).

(*) Supported in part by the Air Force Office of Scientific Research, Air Research and Development Command.

(1) See, for example, H. Primakoff: review talk at Conference on Weak Interaction, Gatlinburg, Tenn. (1958), to be published in *Rev. Mod. Phys.*; J. C. Sens: *Ph. D. Thesis* (University of Chicago (1958); *Phys. Rev.* **113**, 679 (1959); J. C. Sens, R. A. Swanson, V. L. Telegdi and D. D. Yovanovitch: *Phys. Rev.*, **107**, 1464 (1957); R. D. Sard and M. F. Crouch: *Progr. Cosmic Ray Phys.*, **2**, 1 (1954).

The most significant results of these investigations are well known: i) the approximate equality of the effective coupling constants for the reaction $\mu^- + p \to n + \nu$ and the reaction $n \to p + e^- + \bar{\nu}$; ii) the comparatively low mean excitation energy of the daughter nuclei formed in the capture process ((10 ÷ 20) MeV). However nearly all of the studies in question are concerned with the *total* muon capture rate, *i.e.* with the sum of the partial muon capture rates to all energetically accessible bound and unbound levels of the daughter nucleus.

The first investigation in which a *partial* muon capture rate was determined is due to GODFREY [2]. GODFREY studied experimentally the rate of that muon capture reaction

$$\mu^- + {}^{12}_{6}\mathrm{C} \to {}^{12}_{5}\mathrm{B} + \nu\,,$$

which was followed by the β decay of ${}^{12}_{5}\mathrm{B}$

$${}^{12}_{5}\mathrm{B} \to {}^{12}_{6}\mathrm{C} + e^- + \bar{\nu}$$

and accordingly obtained the partial rate of muon capture to all the bound states of ${}^{12}_{5}\mathrm{B}$. He further gave a qualitative argument in favour of the view that most of the muon capture transitions to the bound states of ${}^{12}_{5}\mathrm{B}$ actually go to the ground state of ${}^{12}_{5}\mathrm{B}$; as a result he identified his *observed* partial muon capture rate with the rate from ${}^{12}_{6}\mathrm{C}$ to the ${}^{12}_{5}\mathrm{B}$ ground state. GODFREY then established an approximate theoretical relation between the nuclear matrix elements for muon capture and for β decay between the ground states of ${}^{12}_{6}\mathrm{C}$ and ${}^{12}_{5}\mathrm{B}$; using this relation and comparing his observed muon capture rate with the known ${}^{12}_{5}\mathrm{B}$ β decay rate he concluded that the Gamow-Teller coupling constants in muon capture and in β decay are approximately equal.

In the present paper we reexamine and refine the relation between the nuclear matrix elements for muon capture and for β decay between the ground states of ${}^{12}_{6}\mathrm{C}$ and ${}^{12}_{5}\mathrm{B}$, and also extend the argument to the calculation of the ground state to ground state partial muon capture rates in the reactions

$$\mu^- + {}^{6}_{3}\mathrm{Li} \to {}^{6}_{2}\mathrm{He} + \nu\,,$$

$$\mu^- + {}^{3}_{2}\mathrm{He} \to {}^{3}_{1}\mathrm{H} + \nu\,.$$

For this purpose we use a muon-nucleon effective Hamiltonian developed in the next section.

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(2) T. N. K. GODFREY: *Ph. D. Thesis* (Princeton University, 1954), and *Phys. Rev.*, **92**, 512 (1953).

2. – Effective hamiltonian.

The most general Lorentz covariant form of the transition matrix element, M.E.$^{(\mu)}$, for the reaction $\mu^- + p \to n + \nu$ in a theory where the lepton-*bare* nucleon coupling is V and A, is (³),

$$(1a)\qquad \text{M.E.}^{(\mu)} \quad \frac{1}{\sqrt{2}}\{(\bar{u}_\nu(1-\gamma_5)i\gamma_\lambda\gamma_5 u_\mu)[A^{(\mu)}(\bar{u}_n i\gamma_\lambda\gamma_5 u_p) - B^{(\mu)}(\bar{u}_n(p_\lambda - n_\lambda)\gamma_5 u_p)] + \\ + (\bar{u}_\nu(1-\gamma_5)\gamma_\lambda u_\mu)[C^{(\mu)}(\bar{u}_n\gamma_\lambda u_p) - iD^{(\mu)}(\bar{u}_n\sigma_{\lambda\rho}(p_\rho - n_\rho)u_p)]\},$$

where u_ν, u_μ, u_n, u_p; ν_λ, μ_λ, n_λ, p_λ are, respectively, spinors and four-momenta associated with the indicated particles,

$$(\gamma_\lambda p_\lambda - im_p)u_p = 0\;; \qquad p_\lambda p_\lambda = -m_p^2\,, \qquad \text{etc.}$$

and $A^{(\mu)}$, $B^{(\mu)}$, $C^{(\mu)}$, $D^{(\mu)}$ are functions of the nucleon four-momentum transfer $(p_\lambda - n_\lambda)^2$. In the limit: dressed nucleon $\to$ bare nucleon, $A^{(\mu)}$, $C^{(\mu)} \to A$, V lepton-bare nucleon coupling constants and $B^{(\mu)}$, $D^{(\mu)} \to 0$. An alternative and completely equivalent expression for M.E.$^{(\mu)}$ is obtained by use of the four-momentum conservation law, $p_\lambda - n_\lambda = \nu_\lambda - \mu_\lambda$ and the above mentioned Dirac equations for u_ν, u_μ, u_p, u_n:

$$(1b)\qquad \text{M.E.}^{(\mu)} = \frac{1}{\sqrt{2}}\Big\{A^{(\mu)}(\bar{u}_\nu(1-\gamma_5)i\gamma_\lambda\gamma_5 u_\mu)\,(\bar{u}_n i\gamma_\lambda\gamma_5 u_p) + \\ + m_\mu B^{(\mu)}(\bar{u}_\nu(1-\gamma_5)\gamma_5 u_\mu)\,(\bar{u}_n\gamma_5 u_p) - \frac{iC^{(\mu)}}{m_p}\,(\bar{u}_\nu(1-\gamma_5)\gamma_\lambda u_\mu)\,(\bar{u}_n p_\lambda u_p) - \\ - \frac{iC^{(\mu)}}{2m_p}\,(\bar{u}_\nu(1-\gamma_5)\gamma_\lambda(\mu_\lambda - \nu_\lambda)u_\mu)\,(\bar{u}_n u_p) + \\ + i\left(\frac{C^{(\mu)}}{2m_p} + D^{(\mu)}\right)(\bar{u}_\nu(1-\gamma_5)\gamma_\lambda(\mu_\rho - \nu_\rho)u_\mu)\,(\bar{u}_n\sigma_{\lambda\rho}u_p)\Big\}\,.$$

In this form it is seen that $m_\mu B^{(\mu)}$ is the coupling constant of the « induced » pseudoscalar interaction (³,⁴), while the term in $D^{(\mu)}$ contains the contributions to M.E.$^{(\mu)}$ arising from the assumption of a « conserved vector current » (⁵) which necessitates a direct pion-lepton interaction $(\pi^+ + \mu^- \to \pi^0 + \nu)$. Compa-

(³) M. L. Goldberger and S. B. Treiman: *Phys. Rev.*, **111**, 355 (1958); see however S. Weinberg: *Phys. Rev.*, **112**, 1375 (1958).

(⁴) L. Wolfenstein: *Nuovo Cimento*, **8**, 882 (1958).

(⁵) R. P. Feynman and M. Gell-Mann: *Phys. Rev.*, **109**, 193 (1958); M. Gell-Mann: *Phys. Rev.*, **111**, 362 (1958); J. Bernstein and R. R. Lewis: *Phys. Rev.*, **112**, 232 (1958).

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rison of the nuclear factor in the muon capture transition matrix element which is associated with the V coupling, $[C^{(\mu)}(\overline{u}_n\gamma_\lambda u_p) - iD^{(\mu)}(\overline{u}_n\sigma_{\lambda\rho}(p_\rho - n_\rho)u_p)]$, with the nuclear factors in the corresponding electromagnetic transition matrix elements [6],

$$[F_1^{(p)}(\overline{u}_{p'}\gamma_\lambda u_p) - iF_2^{(p)}(\overline{u}_{p'}\sigma_{\lambda\rho}(p_\rho - p'_\rho)u_p)]\,, \quad [F_1^{(n)}(\overline{u}_{n'}\gamma_\lambda u_n) - iF_2^{(n)}(\overline{u}_{n'}\sigma_{\lambda\rho}(n_\rho - n'_\rho)u_n)]\,,$$

leads to the relation:

$$\text{(2a)} \quad \frac{D^{(\mu)}((p_\lambda - n_\lambda)^2)}{C^{(\mu)}((p_\lambda - n_\lambda)^2)} = \left.\frac{F_2^{(p)}((p_\lambda - p'_\lambda)^2) - F_2^{(n)}((n_\lambda - n'_\lambda)^2)}{F_1^{(p)}((p_\lambda - p'_\lambda)^2) - F_1^{(n)}((n_\lambda - n'_\lambda)^2)}\right]_{(p_\lambda - p'_\lambda)^2 = (n_\lambda - n'_\lambda)^2 = (p_\lambda - n_\lambda)^2},$$

where $F_1^{(p)}((p_\lambda - p'_\lambda)^2)$, $F_1^{(n)}((n_\lambda - n'_\lambda)^2)$, $F_2^{(p)}((p_\lambda - p'_\lambda)^2)$, $F_2^{(n)}((n_\lambda - n'_\lambda)^2)$ are form factor functions associated with the electric charge density and the anomalous magnetic moment density within the proton and the neutron. Further, it is consistent with the Stanford experiments on the electron scattering from the proton and the deuteron to take [7]:

$$\text{(2b)} \quad \begin{cases} F_1^{(n)}((n_\lambda - n'_\lambda)^2) \cong 0\,; \\ F_1^{(p)}((p_\lambda - p'_\lambda)^2)/e \cong \\ \qquad \cong F_2^{(p)}((p_\lambda - p'_\lambda)^2)\Big/\left(\dfrac{e\mu_p}{2m_p}\right) \cong F_2^{(n)}((n_\lambda - n'_\lambda)^2)\Big]_{(n_\lambda - n'_\lambda)^2 = (p_\lambda - p'_\lambda)^2}\Big/\left(\dfrac{e\mu_n}{2m_p}\right), \end{cases}$$

where e, $\mu_p = 1.793$, $\mu_n = -1.913$ are the proton electric charge and the proton, neutron (static) anomalous magnetic moments (in units of $e/2m_p$; $\hbar = 1$, $c = 1$). Thus, from eqs. (2a), (2b),

$$\text{(2c)} \quad \frac{D^{(\mu)}((p_\lambda - n_\lambda)^2)}{C^{(\mu)}((p_\lambda - n_\lambda)^2)} \cong \frac{\mu_p - \mu_n}{2m_p}\,.$$

The « universality » of the muon-bare nucleon and electron-bare nucleon coupling constants now implies

$$\text{(3a)} \quad \begin{cases} A^{(\mu)}((p_\lambda - n_\lambda)^2) = A^{(\beta)}((p_\lambda - n_\lambda)^2)\,; \quad B^{(\mu)}((p_\lambda - n_\lambda)^2) = B^{(\beta)}((p_\lambda - n_\lambda)^2)\,; \\ C^{(\mu)}((p_\lambda - n_\lambda)^2) = C^{(\beta)}((p_\lambda - n_\lambda)^2)\,; \quad D^{(\mu)}((p_\lambda - n_\lambda)^2) = D^{(\beta)}((p_\lambda - n_\lambda)^2)\,, \end{cases}$$

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(6) J. Bernstein and M. L. Goldberger: *Rev. Mod. Phys.*, **30**, 465 (1958); L. L. Foldy: *Rev. Mod. Phys.*, **30**, 471 (1958), and *Phys. Rev.*, **87**, 688 (1952); G. Salzman: *Phys. Rev.*, **99**, 973 (1955).

(7) R. Hofstadter: *Ann. Rev. Nucl. Sci.*, **7**, 231 (1957); R. Hofstadter, F. Bumiller and M. R. Yearian: *Rev. Mod. Phys.*, **30**, 482 (1958).

where $A^{(\beta)}, \ldots, D^{(\beta)}$ are functions of $(p_\lambda - n_\lambda)^2$ which enter into the β decay transition matrix element, M.E.$^{(\beta)}$, in the same way that $A^{(\mu)}, \ldots, D^{(\mu)}$ enter into M.E.$^{(\mu)}$ and where the equalities hold for all values of $(p_\lambda - n_\lambda)^2$. At nucleon four-momentum transfers appropriate to muon capture:

$$(p_\lambda - n_\lambda)^2 \simeq m_\mu^2(1 - m_\mu/m_n) = 0.9\, m_\mu^2 \,,$$

the quantities

$$A^{(\mu)}(0.9\, m_\mu^2)\,, \qquad C^{(\mu)}(0.9\, m_\mu^2)\,, \qquad m_\mu B^{(\mu)}(0.9\, m_\mu^2)$$

are to be identified with the axial vector, vector and « induced » pseudoscalar coupling constants effective in muon capture, *i.e.*,

$$(3b) \qquad g_A^{(\mu)} \equiv A^{(\mu)}(0.9\, m_\mu^2)\,; \quad g_V^{(\mu)} \equiv C^{(\mu)}(0.9\, m_\mu^2)\,; \quad g_P^{(\mu)} \equiv m_\mu B^{(\mu)}(0.9\, m_\mu^2)\,.$$

Similarly, at nucleon four-momentum transfers appropriate to β-decay: $(p_\lambda - n_\lambda)^2 \simeq 0$, the quantities $A^{(\beta)}(0)$, $C^{(\beta)}(0)$ are to be identified with the axial vector and vector coupling constants effective in β decay, *i.e.*

$$(3c) \qquad g_A^{(\beta)} \equiv A^{(\beta)}(0)\,; \qquad g_V^{(\beta)} \equiv C^{(\beta)}(0)\,.$$

Moreover, a dispersion theoretic argument (3) indicates that

$$(3d) \qquad \frac{A^{(\mu)}(0.9 m_\mu^2)}{A^{(\mu)}(0)} \simeq 1 - \frac{1}{\pi}\,\frac{(0.9 m_\mu^2)}{4 m_p^2}\,; \quad \frac{m_\mu B^{(\mu)}(0.9 m_\mu^2)}{A^{(\mu)}(0)} \simeq 8\,,$$

while the assumption of a « conserved vector current » and the first of eq. (2b) imply

$$(3e) \qquad \frac{C^{(\mu)}(0.9 m_\mu^2)}{C^{(\mu)}(0)} = \frac{F_1^{(p)}((p_\lambda - p'_\lambda)^2)]_{(p_\lambda - p'_\lambda)^2 = 0.9 m_\mu^2}}{F_1^{(p)}((p_\lambda - p'_\lambda)^2)]_{(p_\lambda - p'_\lambda)^2 = 0}} \simeq 1 - \frac{1}{6}\,(0.9 m_\mu^2)\,\langle r^2\rangle_{pr}\,,$$

where $\langle r^2\rangle_{pr}$, the mean square radius of the proton's electric charge density, is $\simeq (0.8\cdot 10^{-13}\text{ cm})^2 = ((1/1.75)(1/m_\pi))^2$. Eqs. (3a)–(3e) yield:

$$(3f) \quad \frac{g_A^{(\mu)}}{g_A^{(\beta)}} \simeq 1 - \frac{1}{\pi}\,\frac{(0.9 m_\mu^2)}{4 m_p^2} = 0.999\,; \quad \frac{g_V^{(\mu)}}{g_V^{(\beta)}} \simeq 1 - \frac{1}{6}\,\frac{(0.9 m_\mu^2)}{(1.75 m_\pi)^2} = 0.972\,; \quad \frac{g_P^{(\mu)}}{g_A^{(\beta)}} \simeq 8\,,$$

The transition matrix element in eq. (1b), calculated in a non-relativistic approximation for the u_p, u_n, u_μ, and with $D^{(\mu)}$ expressed in terms of $C^{(\mu)}$ via eq. (2c) and $A^{(\mu)}$, $m_\mu B^{(\mu)}$, $C^{(\mu)}$ given via eq. (3b), corresponds to a suitably chosen non-relativistic effective Hamiltonian, $\mathcal{H}_{\text{eff}}^{(\mu)}$. This $\mathcal{H}_{\text{eff}}^{(\mu)}$, generalized to a many

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nucleon problem, is related to the corresponding M.E.$^{(\mu)}$ by:

$$\text{M.E.}^{(\mu)} = \left\langle \begin{matrix}\text{final state}\\ \text{of } A \text{ nucleons}\end{matrix} \middle| \mathscr{H}_{\text{eff}}^{(\mu)} \middle| \begin{matrix}\text{initial state}\\ \text{of } A \text{ nucleons}\end{matrix} \right\rangle$$

and describes muon capture by an aggregate of A dressed nucleons whose individual virtual-pion fields are however not supposed to perturb each other appreciably [8]. We find for $\mathscr{H}_{\text{eff}}^{(\mu)}$, in a configuration space representation:

$$\mathscr{H}_{\text{eff}}^{(\mu)} = \frac{1}{\sqrt{2}}\,\tau^{(+)}\,\frac{(1-\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\nu}})}{\sqrt{2}}\sum_{i=1}^{A}\tau_i^{(-)}\left\{g_V^{(\mu)}\left(1+\frac{\nu}{2m_p}\right)1\cdot 1_i + \left(g_A^{(\mu)} - g_V^{(\mu)}(1+\mu_p-\mu_n)\frac{\nu}{2m_p}\right)\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}_i - \left(g_P^{(\mu)} - g_A^{(\mu)} - g_V^{(\mu)}(1+\mu_p-\mu_n)\right)\cdot \frac{\nu}{2m_p}\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\nu}}\boldsymbol{\sigma}_i\cdot\hat{\boldsymbol{\nu}} - g_V^{(\mu)}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\nu}}\boldsymbol{\sigma}\cdot\boldsymbol{p}_i/m_p - g_A^{(\mu)}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\nu}}\boldsymbol{\sigma}_i\cdot\boldsymbol{p}_i/m_p\right\}\delta(\boldsymbol{r}-\boldsymbol{r}_i)\,. \tag{4}$$

In eq. (4), $\boldsymbol{\nu} \equiv \nu\hat{\boldsymbol{\nu}}$ and $\boldsymbol{p}_i$ are the three-momenta of the neutrino and the i-th nucleon; 1, 1_i and $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}_i$ are 2×2 matrix unit operators and spin angular momentum operators for the lepton and the i-th nucleon; $\boldsymbol{r}$ and $\boldsymbol{r}_i$ are space co-ordinates for the lepton and the i-th nucleon; $\tau^{(+)}$, $\tau_i^{(-)}$ are isobaric-spin operators which transform a lepton muon state into a lepton neutrino state and an i-th nucleon proton state into an i-th nucleon neutron state; the factor $1/\sqrt{2}$ arises from the normalization of u_ν in eq. (1b); the muon and the nucleons are here treated non-relativistically. This $\mathscr{H}_{\text{eff}}^{(\mu)}$ includes however all first order nucleon recoil and nucleon velocity corrections *i.e.*, includes all terms $\sim \nu/m_p$ and $\sim p_i/m_p$.

With the $\mathscr{H}_{\text{eff}}^{(\mu)}$ of eq. (4), we can obtain the square of the muon capture transition matrix element, $\overline{|\text{M.E.}^{(\mu)}|^2}$, summed over all spin orientations of the neutrino and the daughter nucleus, averaged over all spin orientations of the muon and the parent nucleus, and integrated over all directions of emission

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(8) The actual presence of such perturbations is expected to adjoin « many body » terms to $\mathscr{H}_{\text{eff}}^{(\mu)}$ arising from the possibility of exchange of virtual pions among the nucleons and depending on the relative space coordinates of pairs, triplets, ..., of nucleons. These « many body » terms have been discussed for the analogous $\mathscr{H}_{\text{eff}}^{(\beta)}$ arising in β-decay by J. S. BELL and R. J. BLIN-STOYLE: *Nucl. Phys.*, **6**, 87 (1958); R. J. BLIN-STOYLE, V. GUPTA and H. PRIMAKOFF: to be published in *Nucl. Phys.*; J. FUJITA, Z. MATUMOTO, E. KUROBOSHI and H. MIAZAWA: *Progr. Theor. Phys.*, **20**, 308 (1958); and appear to give rise to corrections of no more than about 10%. One anticipates further that the relative importance of these « many body » terms is about the same in muon capture as in β-decay so that their effect is not only fairly small to begin with but will pretty well cancel out from the ratio of the muon capture and β-decay nuclear matrix elements which we consider: cfr. eqs. (10), (11a); (24)-(29); (36)-(41); (52)-(57), below.

of the neutrino. A straightforward calculation gives:

$$\overline{|\text{M.E.}^{(\mu)}|^2} = \frac{1}{(2\pi)^3}\left(\frac{1}{\pi}\left(\frac{Zm'_\mu}{137}\right)^3\right)\left(\frac{1}{2}\cdot\frac{1}{2J_a+1}\right)\overline{|M^{(\mu)}_{\text{nucl}}(a\to b)|^2}, \tag{5a}$$

where $m'_\mu = m_\mu/(1+(m_\mu/Am_p))$ is the muon reduced mass in the parent μ-mesic atom and the nuclear matrix element is expressed as:

$$\begin{aligned}\overline{|M^{(\mu)}_{\text{nucl}}(a\to b)|^2} = \int\frac{d\hat{\nu}}{4\pi}\sum_{M_b,M_a}\Big\{&(G^{(\mu)}_V)^2|\langle M_b|\sum_i\tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)|M_a\rangle|^2+\\ &+(G^{(\mu)}_A)^2|\langle M_b|\sum_i\tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)\boldsymbol{\sigma}_i|M_a\rangle|^2+\\ &+((G^{(\mu)}_P)^2-2G^{(\mu)}_PG^{(\mu)}_A)|\langle M_b|\sum_i\tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)\boldsymbol{\sigma}_i\cdot\hat{\boldsymbol{\nu}}|M_a\rangle|^2-\\ &-(G^{(\mu)}_V)(g^{(\mu)}_V)\Big[\langle M_b|\sum_i\tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)|M_a\rangle^*\cdot\\ &\cdot\langle M_b|\sum_i\tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)\frac{\boldsymbol{p}_i}{m_p}\cdot\hat{\boldsymbol{\nu}}|M_a\rangle+\text{c.c.}\Big]-\\ &-(G^{(\mu)}_Ag^{(\mu)}_A-G^{(\mu)}_Pg^{(\mu)}_A)\Big[\langle M_b|\sum_i\tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)\boldsymbol{\sigma}_i\cdot\hat{\boldsymbol{\nu}}|M_a\rangle^*\cdot\\ &\langle M_b|\sum_i\tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)\boldsymbol{\sigma}_i\cdot\frac{\boldsymbol{p}_i}{m_p}|M_a\rangle+\text{c.c.}\Big]+\\ &+[\text{relatively negligible terms in }(p_i/m_p)^2]\Big\},\end{aligned} \tag{5b}$$

with

$$\begin{cases} G^{(\mu)}_V \equiv g^{(\mu)}_V\left(1+\dfrac{\nu}{2m_p}\right); \qquad G^{(\mu)}_A \equiv g^{(\mu)}_A - g^{(\mu)}_V(1+\mu_p-\mu_n)\dfrac{\nu}{2\,m_p};\\ G^{(\mu)}_P \equiv (g^{(\mu)}_P - g^{(\mu)}_A - g^{(\mu)}_V(1+\mu_p-\mu_n))\dfrac{\nu}{2m_p}. \end{cases} \tag{5c}$$

The corresponding calculation for the square of the β decay transition matrix element, $\overline{|\text{M.E.}^{(\beta)}|^2}$, in which the roles of the parent and daughter nuclei are interchanged, yields for allowed transitions:

$$\overline{|\text{M.E.}^{(\beta)}|^2} = \frac{1}{(2\pi)^6}F(Z,E)\frac{1}{2J_b+1}\overline{|M^{(\beta)}_{\text{nucl}}(b\to a)|^2}, \tag{6a}$$

where $F(Z,E)$ is the Fermi function for an emitted electron energy E and the nuclear matrix element is given by:

$$\overline{|M^{(\beta)}_{\text{nucl}}(b\to a)|^2} = \sum_{M_b\cdot M_a}\{(g^{(\beta)}_V)^2|\langle M_a|\sum_i\tau_i^{(+)}|M_b\rangle|^2+(g^{(\beta)}_A)^2|\langle M_a|\sum_i\tau_i^{(+)}\boldsymbol{\sigma}_i|M_b\rangle|^2\}. \tag{6b}$$

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In Eqs. (5a), (5b) and (6a), (6b), $|M_a\rangle$, $|M_b\rangle$ represent wavefunctions of the two nuclear states involved; these wavefunctions are characterized by energy, parity, spin, and spin orientation or magnetic quantum numbers E_a, $\mathcal{P}_a$, J_a, M_a; E_b, $\mathcal{P}_b$, J_b, M_b. The quantity $\varphi(r_i)$ is the muon space orbital wavefunction taken at the position of the i-th nucleon and normalized in such a way that $\varphi(r_i)\to 1$ as $Z\to 0$, *i.e.* for small Z, $\varphi(r_i)\simeq\varphi_Z(0)\exp[-(Zm'_\mu/137)r_i]$ with $\varphi_Z(0)\to 1$ as $Z\to 0$; we take $\varphi_Z(0)\simeq 1$, for the light nuclei that we consider, as an approximation additional to that contained in the assumption of the exponential dependence of $\varphi(r_i)$ on r_i.

It may now be remarked that in evaluating $\overline{|\text{M.E.}^{(\mu)}|^2}$ one may write in the third term of eq. (5b):

$$
\begin{aligned}
(7)\qquad &\int\frac{d\hat{\boldsymbol{\nu}}}{4\pi}\sum_{M_b,M_a}\Big|\langle M_b|\sum_i\tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)\boldsymbol{\sigma}_i|M_a\rangle\cdot\hat{\boldsymbol{\nu}}\Big|^2=\\
&=\int\frac{d\hat{\boldsymbol{\nu}}}{4\pi}\sum_{M_b,M_a}\Big|\langle M_b|\sum_i\tau_i^{(-)}\sum_l j_l(\nu r_i)\sqrt{4\pi(2l+1)}\,(i)^{-l}\,Y_{l,0}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{r}}_i)\varphi(r_i)\boldsymbol{\sigma}_i|M_a\rangle\cdot\hat{\boldsymbol{\nu}}\Big|^2=\\
&=\int\frac{d\hat{\boldsymbol{\nu}}}{4\pi}\sum_{M_b,M_a}\sum_l\Big|\langle M_b|\sum_i\tau_i^{(-)}j_l(\nu r_i)\sqrt{4\pi(2l+1)}\,Y_{l,0}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{r}}_i)\varphi(r_i)\boldsymbol{\sigma}_i|M_a\rangle\cdot\hat{\boldsymbol{\nu}}\Big|^2,
\end{aligned}
$$

where j_l and $Y_{l,0}$ are spherical Bessel and spherical harmonic functions, respectively, and where the last equality is justified since there is no interference in the overall muon capture rate and so none in $\overline{|M^{(\mu)}_{\text{nucl}}(a\to b)|^2}$ between emitted neutrinos with different l. Moreover for the muon capture transitions considered in this paper: $\mu^-+{}^3_2\text{He}\to{}^3_1\text{H}+\nu$; $\mu^-+{}^6_3\text{Li}\to{}^6_2\text{He}+\nu$; $\mu^-+{}^{12}_6\text{C}\to$ $\to{}^{12}_5\text{B}+\nu$, one has: $J_a=J_b=\frac{1}{2}$, $\mathcal{P}_a=\mathcal{P}_b$; $J_a=1$, $J_b=0$, $\mathcal{P}_a=\mathcal{P}_b$; $J_a=0$, $J_b=1$, $\mathcal{P}_a=\mathcal{P}_b$ so that only terms with $l=0$, 2 survive in eq. (7), *i.e.* only s-wave and d-wave neutrinos are emitted. In addition, explicit estimates, such as in eqs. (48)–(52) below, indicate that at least for light nuclei the d-wave contribution is relatively small so that:

$$
\begin{aligned}
(8)\qquad &\int\frac{d\hat{\boldsymbol{\nu}}}{4\pi}\sum_{M_b,M_a}\Big|\langle M_b|\sum_i\tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)\boldsymbol{\sigma}_i|M_a\rangle\cdot\hat{\boldsymbol{\nu}}\Big|^2\cong\\
&\cong\int\frac{d\hat{\boldsymbol{\nu}}}{4\pi}\sum_{M_b,M_a}\Big|\langle M_b|\sum_i\tau_i^{(-)}j_0(\nu r_i)\,\varphi(r_i)\boldsymbol{\sigma}_i|M_a\rangle\cdot\hat{\boldsymbol{\nu}}\Big|^2=\\
&=\frac{1}{3}\int\frac{d\hat{\boldsymbol{\nu}}}{4\pi}\sum_{M_b,M_a}\Big|\langle M_b|\sum_i\tau_i^{(-)}j_0(\nu r_i)\varphi(r_i)\boldsymbol{\sigma}_i|M_a\rangle\Big|^2\cong\\
&\cong\frac{1}{3}\int\frac{d\hat{\boldsymbol{\nu}}}{4\pi}\sum_{M_b,M_a}\Big|\langle M_b|\sum_i\tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)\boldsymbol{\sigma}_i|M_a\rangle\Big|^2.
\end{aligned}
$$

Also, the last two terms in eq. (5b) which are $\sim p_i/m_p$ may be estimated to be no more than a few percent of the first three terms—this conclusion is

supported by β decay data which shows that nuclear matrix elements $\sim p_i/m_p$ correspond to $ft_{\frac{1}{2}}$ values which are some $(10^3 \div 10^4)$ times greater than the $ft_{\frac{1}{2}}$ values for superallowed transitions. Thus from eqs. (5b), (8) and with neglect of the p_i/m_p terms:

$$(9a)\qquad \overline{|M_{\text{nucl}}^{(\mu)}(a\to b)|^2} = \int \frac{\mathrm{d}\hat{\boldsymbol{\nu}}}{4\pi} \sum_{M_b, M_a} \{ (G_V^{(\mu)})^2 |\langle M_b | \sum \tau_i^{(-)} \exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i) | M_a\rangle|^2 + \\ + (\Gamma_A^{(\mu)})^2 |\langle M_b | \sum_i \tau_i^{(-)} \exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)\boldsymbol{\sigma}_i | M_a\rangle|^2 \},$$

where

$$(9b)\qquad (\Gamma_A^{(\mu)})^2 \equiv (G_A^{(\mu)})^2 + \tfrac{1}{3}((G_P^{(\mu)})^2 - 2G_P^{(\mu)}G_A^{(\mu)}) .$$

3. – General considerations.

The ratio of the transition rates of the muon-capture from $|M_a\rangle$ to $|M_b\rangle$ and the β decay from $|M_b\rangle$ to $|M_a\rangle$, $w^{(\mu)}/w^{(\beta)}$, becomes, taking proper account of the densities of final states in the two cases and using eqs. (5a) and (6a):

$$(10)\qquad \frac{w^{(\mu)}}{w^{(\beta)}} = \left[\pi\nu^2\left(1 - \frac{\nu}{m_\mu + Am_p}\right)\left(\frac{Zm'_\mu}{137}\right)^3 \frac{1}{f}\right]\cdot \\ \cdot\left(\frac{2J_b+1}{2J_a+1}\right)\cdot\left(\frac{\overline{|M_{\text{nucl}}^{(\mu)}(a\to b)|^2}}{\overline{|M_{\text{nucl}}^{(\beta)}(b\to a)|^2}}\right) \equiv k\cdot\left(\frac{2J_b+1}{2J_a+1}\right)\cdot R .$$

In eq. (10), ν is obtained by the conservation of energy and momentum as

$$\nu = \left[m_\mu\left(1 - \frac{1}{2}\left(\frac{Z}{137}\right)^2\right) - E_{\max}\right]\left(1 - \frac{m_\mu}{2(m_\mu + Am_p)}\right),$$

and f is the usual integral over the electron spectrum of β decay theory:

$$f(Z, E_{\max}) = \int_1^{E_{\max}} F(Z, E)(E_{\max} - E)^2 E\sqrt{E^2 - 1}\, \mathrm{d}E ,$$

R is given by eqs. (10), (9a), (9b) and (6b) as:

$$(11a)\qquad R = \frac{(G_V^{(\mu)})^2}{(g_V^{(\beta)})^2}\left(\frac{\int(\mathrm{d}\hat{\boldsymbol{\nu}}/4\pi)\sum_{M_b,M_a}\{|\langle M_b|\sum_i \tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)|M_a\rangle|^2}{\sum_{M_b,M_a}\{|\langle M_b|\sum_i \tau_i^{(-)}|M_a\rangle|^2 + (g_A^{(\beta)}/g_V^{(\beta)})^2|\langle M_b|\sum_i \tau_i^{(-)}\boldsymbol{\sigma}_i|M_a\rangle|^2\}} + \right. \\ \left. + \frac{\int(\mathrm{d}\hat{\boldsymbol{\nu}}/4\pi)(\Gamma_A^{(\mu)}/G_V^{(\mu)})^2|\langle M_b \sum_i \tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)\boldsymbol{\sigma}_i|M_a\rangle|^2\}}{\sum_{M_b,M_a}\{|\langle M_b|\sum_i \tau_i^{(-)}|M_a\rangle|^2 + (g_A^{(\beta)}/g_V^{(\beta)})^2|\langle M_b|\sum_i \tau_i^{(-)}\boldsymbol{\sigma}_i|M_a\rangle|^2\}}\right),$$

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with (see eqs. (5c), (3f), (9b))

$$
(11b)\quad \begin{cases}
\left(\dfrac{G_V^{(\mu)}}{g_V^{(\beta)}}\right)^2 = (0.97)^2(1+\nu/2m_p)^2\,; \\[2ex]
\left(\dfrac{\Gamma_A^{(\mu)}}{G_V^{(\mu)}}\right)^2 = \left(\dfrac{g_A^{(\beta)}}{g_V^{(\beta)}}\right)^2 \cdot \\[2ex]
\cdot\left\{\dfrac{[1-0.97(g_V^{(\beta)}/g_A^{(\beta)})4.71(\nu/2m_p)]^2+\frac{1}{3}\,[(7-0.97)(g_V^{(\beta)}/g_A^{(\beta)})\cdot 4.71)\,(\nu/2m_p)]^2}{(0.97)^2(1+\nu/2m_p)^2} - \right. \\[2ex]
\left. -\dfrac{\frac{2}{3}[(7-0.97(g_V^{(\beta)}/g_A^{(\beta)})\cdot 4.71)\,(\nu/2m_p)\,(1-0.97(g_V^{(\beta)}/g_A^{(\beta)})\cdot 4.71(\nu/2m_p))]}{(0.97)^2(1+\nu/2m_p)^2}\right\}.
\end{cases}
$$

Now the values of ν are 91.4, 100.7, 103.3 (in MeV/c) for ${}^{12}_{6}\mathrm{C}\to{}^{12}_{5}\mathrm{B}$, ${}^{6}_{3}\mathrm{Li}\to{}^{6}_{2}\mathrm{He}$, ${}^{3}_{2}\mathrm{He}\to{}^{3}_{1}\mathrm{H}$, respectively, while analysis of β decay experiments shows that $1.18\leqslant-(g_A^{(\beta)}/g_V^{(\beta)})\leqslant 1.25$ [9]. The value of the quantity in the curly brackets in eq. (11b) is rather insensitive to the exact magnitude of $-(g_A^{(\beta)}/g_V^{(\beta)})$ and we take $-(g_A^{(\beta)}/g_V^{(\beta)})=1.21$ to evaluate it. We thus obtain:

$$
(11c)\quad \begin{cases}
\left(\dfrac{G_V^{(\mu)}}{g_V^{(\beta)}}\right)^2 = \begin{cases} 1.03:\ {}^{12}_{6}\mathrm{C}\to{}^{12}_{5}\mathrm{B} \\ 1.04:\ {}^{6}_{3}\mathrm{Li}\to{}^{6}_{2}\mathrm{He};\ {}^{3}_{2}\mathrm{He}\to{}^{3}_{1}\mathrm{H}\,; \end{cases} \\[3ex]
\left(\dfrac{\Gamma_A^{(\mu)}}{G_V^{(\mu)}}\right)^2 = \left(\dfrac{g_A^{(\beta)}}{g_V^{(\beta)}}\right)^2\cdot\begin{pmatrix} 1.05:\ {}^{12}_{6}\mathrm{C}\to{}^{12}_{5}\mathrm{B} \\ 1.04:\ {}^{6}_{3}\mathrm{Li}\to{}^{6}_{2}\mathrm{He};\quad {}^{3}_{2}\mathrm{He}\to{}^{3}_{1}\mathrm{H}\end{pmatrix}; \\[3ex]
\left(\dfrac{\Gamma_A^{(\mu)}}{g_A^{(\beta)}}\right)^2 = \dfrac{(\Gamma_A^{(\mu)}/G_V^{(\mu)})^2}{(g_A^{(\beta)}/g_V^{(\beta)})^2}\cdot\left(\dfrac{G_V^{(\mu)}}{g_V^{(\beta)}}\right)^2 = 1.08:\ {}^{12}_{6}\mathrm{C}\to{}^{12}_{5}\mathrm{B};\ {}^{6}_{3}\mathrm{Li}\to{}^{6}_{2}\mathrm{He};\ {}^{3}_{2}\mathrm{He}\to{}^{3}_{1}\mathrm{H}\,.
\end{cases}
$$

Deviations from « universality » between $g_A^{(\mu)}$, $g_V^{(\mu)}$ and $g_A^{(\beta)}$, $g_V^{(\beta)}$ or from $g_P^{(\mu)}/g_A^{(\beta)}\simeq 8$ (eq. (3f)), or absence of the anomalous nucleon magnetic moment term $\sim(\mu_p-\mu_n)$ associated with the « conserved vector current » (eqs. (2c), (4), (5c)) will manifest themselves as deviations of $(G_V^{(\mu)}/g_V^{(\beta)})^2$, $(\Gamma_A^{(\mu)}/G_V^{(\mu)})^2$, and $(\Gamma_A^{(\mu)}/g_A^{(\beta)})^2$ from the values given in eq. (11c).

We now describe an approximate but widely applicable method for finding R. We have as a general relation between corresponding muon capture and β decay nuclear matrix elements:

$$
(12a)\quad \begin{cases}
\langle M_b|\sum_i \tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)|M_a\rangle = \\
\qquad\qquad = \langle M_b|\sum_i \tau_i^{(-)}|M_a\rangle\displaystyle\int\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}]\varphi(r)D_{M_b,M_a}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}\,, \\[2ex]
\langle M_b|\sum_i \tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)\sigma^{(k)}M_a\rangle = \\
\qquad\qquad = \langle M_b|\sum_i \tau_i^{(-)}\sigma_i^{(k)}|M_a\rangle\displaystyle\int\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}]\varphi(r)D^{\sigma(k)}_{M_b,M_a}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}\,,
\end{cases}
$$

(9) C. S. Wu and V. L. Telegdi: review talks at Conference on Weak Interactions, Gatlinburg, Tenn. (1958), to be published in *Rev. Mod. Phys.*

where the «off-diagonal» density distribution functions $D_{M_b,M_a}(\boldsymbol{r})$ and $D^{\sigma(k)}_{M_b,M_a}(\boldsymbol{r})$ are:

$$(12b)\quad \begin{cases} D_{M_b,M_a}(\boldsymbol{r}) = \dfrac{\langle M_b|\sum_i \tau_i^{(-)}\delta(\boldsymbol{r}-\boldsymbol{r}_i)|M_a\rangle}{\langle M_b|\sum_i \tau_i^{(-)}|M_a\rangle}, \\[2ex] D^{\sigma(k)}_{M_b,M_a}(\boldsymbol{r}) = \dfrac{\langle M_b|\sum_i \tau_i^{(-)}\sigma_i^{(k)}\,\delta(\boldsymbol{r}-\boldsymbol{r}_i)|M_a\rangle}{\langle M_b|\sum_i \tau_i^{(-)}\sigma_i^{(k)}|M_a\rangle}, \qquad (k=1,2,3), \end{cases}$$

while the (directionally-averaged-over) density distribution function of the protons in the parent nucleus, $\mathcal{D}_a(r)$, is

$$(13)\qquad \mathcal{D}_a(r) = \frac{1}{2J_a+1}\sum_{M_a}\frac{\langle M_a|\sum_i \frac{1}{2}(1+\tau_i^{(3)})\delta(\boldsymbol{r}-\boldsymbol{r}_i)|M_a\rangle}{\langle M_a|\sum_i \frac{1}{2}(1+\tau_i^{(3)})|M_a\rangle}.$$

It is important to note that for the $\mu^- + {}^6_3\mathrm{Li} \to {}^6_2\mathrm{He} + \nu$ and $\mu^- + {}^{12}_6\mathrm{C} \to {}^{12}_5\mathrm{B} + \nu$ transitions where $J_a=1$, $J_b=0$, $\mathcal{P}_a=\mathcal{P}_b=1$, and $J_a=0$, $J_b=1$, $\mathcal{P}_a=\mathcal{P}_b=1$, respectively, the vanishing of $\langle M_b|\sum_i \tau_i^{(-)}|M_a\rangle$ in the denominator of $D_{M_b,M_a}(\boldsymbol{r})$ is accompanied by the vanishing of $\langle M_b|\sum_i \tau_i^{(-)}\delta(\boldsymbol{r}-\boldsymbol{r}_i)|M_a\rangle$ in the corresponding numerator so that $D_{M_b,M_a}(\boldsymbol{r})$ is finite and $\langle M_b|\sum_i \tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)|M_a\rangle = 0$. This follows since, *e.g.*,

$$\langle \mathcal{P}_b=1, J_b=1, M_b|\sum_i \tau_i^{(-)}\delta(\boldsymbol{r}-\boldsymbol{r}_i)|\mathcal{P}_a=1, J_a=0, M_a\rangle =$$
$$= \sum_{l,m}\frac{1}{r^2}Y_{l,m}(\theta,\varphi)\langle \mathcal{P}_b=1, J_b=1, M_b|\sum_i \tau_i^{(-)}\delta(r-r_i)Y_{lm}(\theta_i,\varphi_i)|\mathcal{P}_a=1, J_a=0, M_a\rangle,$$

with every

$$\langle \mathcal{P}_b=1, J_b=1, M_b|\sum_i \tau_i^{(-)}\delta(r-r_i)Y_{lm}(\theta_i,\varphi_i)|\mathcal{P}_a=1, J_a=0, M_a\rangle = 0$$

because of the contradictory demands of the $\mathcal{P}$ and J selection rules ($l=0, 2, 4,\ldots$ and $l=1$, respectively).

If now the wavefunctions $|M_a\rangle$, $|M_b\rangle$ are factorizable as:

$$(14a)\quad \begin{cases} |M_a\rangle = u_{E_a,\mathcal{P}_a,J_a}(\ldots,\boldsymbol{r}_i,\ldots)\,v_{J_a,M_a}(\ldots,\sigma_i^{(3)},\tau_i^{(3)},\ldots) = \\ \qquad\qquad = u(\ldots,r_{ij},\ldots)\,v_{J_a,M_a}(\ldots,\sigma_i^{(3)},\tau_i^{(3)},\ldots), \\[1ex] |M_b\rangle = u_{E_b,\mathcal{P}_b,J_b}(\ldots,\boldsymbol{r}_i,\ldots)\,v_{J_b,M_b}(\ldots,\sigma_i^{(3)},\tau_i^{(3)},\ldots) = \\ \qquad\qquad = u(\ldots,r_{ij},\ldots)\,v_{J_b,M_b}(\ldots,\sigma_i^{(3)},\tau_i^{(3)},\ldots), \end{cases}$$

which is quite accurately valid for ${}^3_2$He and ${}^3_1$H (see eqs. (19a)–(20c), below), one has from eqs. (12b), (13):

$$D_{M_b,M_a}(\boldsymbol{r}) = D^{\sigma(k)}_{M_b,M_a}(\boldsymbol{r}) = \mathcal{D}_a(r) = \langle u|\delta(\boldsymbol{r}-\boldsymbol{r}_1)|u\rangle\,. \tag{14b}$$

Under these circumstances the muon capture nuclear matrix elements in eq. (12a) are expressed as products of the corresponding β decay nuclear matrix elements multiplied by,

$$\int \exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}]\varphi(r)\,\mathcal{D}_a(r)\,\mathrm{d}\boldsymbol{r} = \int j_0(\nu r)\varphi(r)\,\mathcal{D}_a(r)\,\mathrm{d}\boldsymbol{r}\,, \tag{14c}$$

a relation which shows that eq. (14a) for $|M_a\rangle$, $|M_b\rangle$ implies that only s-wave neutrinos are emitted. The integral in eq. (14c) is a quantity which for $\varphi(r)\simeq 1$ is just the form factor for electron elastic scattering from the parent nucleus for an electron momentum transfer equal numerically to ν. Since

$$\varphi(r)\simeq\varphi_z(0)\exp\left[-\frac{Zm'_\mu}{137}\,r\right]\simeq\exp\left[-\frac{Zm'_\mu}{137}\,r\right]\simeq 1-\frac{Zm'_\mu}{137}\,r\,,$$

one has in addition:

$$\int j_0((\nu r)\varphi(r)\,\mathcal{D}_a(r)\,\mathrm{d}\boldsymbol{r}\simeq\left(\int j_0(\nu r)\,\mathcal{D}_a(r)\,\mathrm{d}\boldsymbol{r}\right)\cdot\left(1-\frac{Zm'_\mu}{137}\,\frac{\int j_0(\nu r)r\,\mathcal{D}_a(r)\,\mathrm{d}\boldsymbol{r}}{\int j_0(\nu r)\,\mathcal{D}_a(r)\,\mathrm{d}\boldsymbol{r}}\right)\simeq \tag{15a}$$

$$\simeq\left(1-\frac{1}{6}\,(\nu^2\langle r^2\rangle_a)+\frac{\gamma_4}{120}\,(\nu^2\langle r^2\rangle_a)^2+\dots\right)\cdot$$

$$\cdot\left(1-\frac{Zm'_\mu}{137\nu}\,\gamma_1(\nu^2\langle r^2\rangle_a)^{\frac12}\,\frac{1-\frac16\,(\gamma_3/\gamma_1)\,(\nu^2\langle r^2\rangle_a)}{1-\frac16\,(\nu^2\langle r^2\rangle_a)}\right),$$

where

$$\langle r^n\rangle_a\equiv\int r^n\,\mathcal{D}_a(r)\,\mathrm{d}\boldsymbol{r}\equiv\gamma_n(\langle r^2\rangle_a)^{n/2}\,, \tag{15b}$$

the numerical coefficients γ_n depending on the functional form of $\mathcal{D}_a(r)$.

Even though the wavefunctions $|M_a\rangle$, $|M_b\rangle$ are not in general factorizable as in eq. (14a) we may nevertheless expect that the following extension of eq. (14b) still holds approximately, at least for $1p$-shell nuclei like ${}^6_3$Li, ${}^6_2$He and ${}^{12}_6$C, ${}^{12}_5$B:

$$\int\frac{\mathrm{d}\hat{\boldsymbol{\nu}}}{4\pi}\sum_{M_b,M_a}|\langle M_b|\sum_i\tau_i^{(-)}|M_a\rangle|^2\left|\int\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}]\varphi(r)D_{M_b,M_a}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}\right|^2= \tag{16a}$$

$$=\int\frac{\mathrm{d}\hat{\boldsymbol{\nu}}}{4\pi}\sum_{M_b,M_a}|\langle M_b|\sum_i\tau_i^{(-)}|M_a\rangle|^2\left\{\left|\int j_0(\nu r)\varphi(r)D_{M_b,M_a}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}\right|^2+\right.$$

$$\left.+\left|\int j_2(\nu r)\sqrt{4\pi\cdot 3}\,Y_{2,0}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{r}})\varphi(r)D_{M_b,M_a}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}\right|^2+\dots\right\}\simeq$$

$$\simeq\int\frac{\mathrm{d}\hat{\boldsymbol{\nu}}}{4\pi}\left(\sum_{M_b,M_a}|\langle M_b|\sum_i\tau_i^{(-)}|M_a\rangle|^2\right)\cdot\left(\left|\int j_0(\nu r)\varphi(r)\,\mathcal{D}_a(r)\,\mathrm{d}\boldsymbol{r}\right|^2\right),$$

$$(16b)\quad \int\frac{\mathrm{d}\hat{\boldsymbol{\nu}}}{4\pi}\sum_{M_b\cdot M_a}\sum_{k=1}^{3}|\langle M_b|\sum_i\tau_i^{(-)}\sigma_i^{(k)}|M_a\rangle|^2\left|\int\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}]\varphi(r)D^{\sigma(k)}_{M_b\cdot M_a}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}\right|^2=$$

$$=\int\frac{\mathrm{d}\hat{\boldsymbol{\nu}}}{4\pi}\sum_{M_b,M_a}\sum_{k=1}^{3}|\langle M_b|\sum_i\tau_i^{(-)}\sigma_i^{(k)}|M_a\rangle|^2\left\{\left|\int j_0(\nu r)\varphi(r)D^{\sigma(k)}_{M_b\cdot M_a}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}\right|^2+\right.$$

$$\left.+\left|\int j_2(\nu r)\sqrt{4\pi\cdot 3}\,Y_{2,0}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{r}})\varphi(r)D^{\sigma(k)}_{M_a\cdot M_b}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}\right|^2+\ldots\right\}\simeq$$

$$\simeq\int\frac{\mathrm{d}\hat{\boldsymbol{\nu}}}{4\pi}\left(\sum_{M_b\cdot M_a}|\langle M_b|\sum_i\tau_i^{(-)}\boldsymbol{\sigma}_i|M_a\rangle|^2\right)\cdot\left(\left|\int j_0(\nu r)\varphi(r)\mathcal{D}_a(r)\,\mathrm{d}\boldsymbol{r}\right|^2\right).$$

In eqs. ($16a$), ($16b$), as in eq. (7), there is no interference between emitted neutrinos with different l. The wave functions $|M_a\rangle$, $|M_b\rangle$ in these equations are supposed to be characterized by $\mathcal{P}_a=\mathcal{P}_b$; as a result $D_{M_b\cdot M_a}(\boldsymbol{r})=D_{M_b\cdot M_a}(-\boldsymbol{r})$ so that terms associated with the emission of odd l-wave neutrinos are rigorously absent. Further, $\mathcal{D}_a(r)$ is considered in these equations to be the density distribution function of the protons in the $1p$-shell of the parent nucleus rather than the density distribution function of all the protons. This last restriction is a consequence of the fact that $1s$-shell protons may be viewed as essentially inert in the ground state $\leftrightarrow$ ground state muon capture transitions in the $1p$-shell nuclei. The approximation of eqs. ($16a$), ($16b$) is thus seen to involve the neglect of the relatively small probability of emission of d-wave, ..., neutrinos compared to that of s-wave neutrinos (cf. eqs. (48)-(52) below) and the replacement of the spatially spherical part of $D_{M_b\cdot M_a}(\boldsymbol{r})$, $D^{\sigma(k)}_{M_b,M_a}(\boldsymbol{r})$ by $\mathcal{D}_a(r)$.

Eqs. ($11a$), ($12a$), ($16a$), ($16b$), and the use of eq. ($15a$) with the $\mathcal{D}_a(r)$ of eqs. ($16a$), ($16b$), yield:

$$(17a)\quad R\simeq K_1K_2\left(\frac{G_V^{(\mu)}}{g_V^{(\beta)}}\right)^2\frac{\sum\limits_{M_b\cdot M_a}\left\{|\langle M_b|\sum_i\tau_i^{(-)}|M_a\rangle|^2+(\Gamma_A^{(\mu)}/G_V^{(\mu)})^2|\langle M_b|\sum_i\tau_i^{(-)}\boldsymbol{\sigma}_i|M_a\rangle|^2\right\}}{\sum\limits_{M_b\cdot M_a}\left\{|\langle M_b|\sum_i\tau_i^{(-)}|M_a\rangle|^2+(g_A^{(\beta)}/g_V^{(\beta)})^2|\langle M_b|\sum_i\tau_i^{(-)}\boldsymbol{\sigma}_i|M_a\rangle|^2\right\}}$$

where

$$(17b)\qquad K_1\equiv\left(1-\frac{1}{3}(\nu^2\langle r^2\rangle_a)+\frac{1}{36}(\nu^2\langle r^2\rangle_a)^2\left(1+\frac{3}{5}\gamma_4\right)+\ldots\right),$$

$$(17c)\qquad K_2\equiv\left(1-\frac{2Zm'_\mu}{137\nu}\gamma_1(\nu^2\langle r^2\rangle_a)^{\frac{1}{2}}\frac{1-\frac{1}{6}(\gamma_3/\gamma_1)(\nu^2\langle r^2\rangle_a)}{1-\frac{1}{6}(\nu^2\langle r^2\rangle_a)}\right).$$

For the transition $\mu^-+{}^3_2\mathrm{He}\to{}^3_1\mathrm{H}+\nu$, one has to a sufficient accuracy:

$$\tfrac{1}{2}\cdot\sum_{M_b,M_a}|\langle M_b|\sum_i\tau_i^{(-)}|M_a\rangle|^2=\tfrac{1}{2}\cdot\tfrac{1}{3}\cdot\sum_{M_b\cdot M_a}|\langle M_b|\sum_i\tau_i^{(-)}\boldsymbol{\sigma}_i)|M_a\rangle|^2=1$$

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(see eqs. (23b), (23d), below) so that eq. (17a) gives, using also eq. (11c) and $(g_A^{(\beta)}/g_V^{(\beta)})^2 = (1.21)^2$:

$$(18a)\qquad R({}^3_2\mathrm{He} \rightleftarrows {}^3_1\mathrm{H}) \simeq K_1({}^3_2\mathrm{He})K_2({}^3_2\mathrm{He})\left[\left(\frac{G_V^{(\mu)}}{g_V^{(\beta)}}\right)^2 \cdot \frac{1+(\Gamma_A^{(\mu)}/G_V^{(\mu)})^2\cdot 3}{1+(g_A^{(\beta)}/g_V^{(\beta)})^2\cdot 3}\right] = \\ = K_1({}^3_2\mathrm{He})K_2({}^3_2\mathrm{He})\cdot(1.07)\,.$$

For $\mu^- + {}^6_3\mathrm{Li} \to {}^6_2\mathrm{He} + \nu$; $\mu^- + {}^{12}_6\mathrm{C} \to {}^{12}_5\mathrm{B} + \nu$, one has $\langle M_b|\sum_i \tau_i|M_a\rangle = 0$, and, from eqs. (17$a$), (11$c$):

$$(18b)\qquad R({}^6_3\mathrm{Li} \rightleftarrows {}^6_2\mathrm{He}) \simeq K_1({}^6_3\mathrm{Li})\,K_2({}^6_3\mathrm{Li})\left(\frac{\Gamma_A^{(\mu)}}{g_A^{(\beta)}}\right)^2 = K_1({}^6_3\mathrm{Li})\,K_2({}^6_3\mathrm{Li})\cdot(1.08)\,,$$

$$(18c)\qquad R({}^{12}_6\mathrm{C} \rightleftarrows {}^{12}_5\mathrm{B}) \simeq K_1({}^{12}_6\mathrm{C})K_2({}^{12}_6\mathrm{C})\left(\frac{\Gamma_A^{(\mu)}}{g_A^{(\beta)}}\right)^2 = K_1({}^{12}_6\mathrm{C})K_2({}^{12}_6\mathrm{C})\cdot(1.08)\,\cdot$$

4. – Particular examples.

In this Section we shall calculate R adopting specific forms for the nuclear wavefunctions $|M_a\rangle$, $|M_b\rangle$. We shall find an expression for R of the form of eqs. (17a)–(17c) with specific numerical values for γ_4, γ_1, γ_3 appropriate to the calculated shape of $\mathcal{D}_a(r)$. For ${}^{12}_6\mathrm{C}$ there will be an additional slight modification of the numerical coefficient of the $(\nu^2\langle r^2\rangle_a)^2$ term in eqs. (17b), (18c) since in this case $D^{\sigma(k)}_{M_b,M_a}(\boldsymbol{r})$ will possess a spatially non-spherical part associated, as noted after eq. (16b), with the small relative probability of emission of a d-wave neutrino

$$\left(\approx\left|\int j_2(\nu r)\sqrt{4\pi\cdot 3}\;Y_{2,0}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{r}})\varphi(r)\,D^{\sigma(k)}_{M_b,M_a}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}\right|^2\Big/\left|\int j_0(\nu r)\,\varphi(r)\,\mathcal{D}_a(r)\,\mathrm{d}\boldsymbol{r}\right|^2\right).$$

A) ${}^3_2\mathrm{He} \rightleftarrows {}^3_1\mathrm{H}$. – The transition between the ground states is here: $(J_a = \frac{1}{2}, \mathcal{P}_a = +1) \rightleftarrows (J_b = \frac{1}{2}, \mathcal{P}_b = +1)$. Assuming that $|M_a\rangle$, $|M_b\rangle$ are, to a sufficient approximation, spin doublet states: ${}^2S_{\frac{1}{2}}$, the factorization discussed in connection with eq. (14a) is valid and one may write:

$$(19a)\qquad \begin{cases} |M_a\rangle = u_{E_a,\mathcal{P}_a,J_a}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)\,v_{J_a,M_a}(\sigma_1^{(3)}, \sigma_2^{(3)}, \sigma_3^{(3)};\ \tau_1^{(3)}, \tau_2^{(3)}, \tau_3^{(3)})\,,\\ |M_b\rangle = u_{E_b,\mathcal{P}_b,J_b}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)\,v_{J_b,M_b}(\sigma_1^{(3)}, \sigma_2^{(3)}, \sigma_3^{(3)};\ \tau_1^{(3)}, \tau_2^{(3)}, \tau_3^{(3)})\,,\end{cases}$$

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with

$$(19b)\qquad u_{E_a,\mathcal{P}_a,J_a}(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3)\exp[-i\boldsymbol{K}_a\cdot\boldsymbol{R}]=$$
$$=u_{E_b,\mathcal{P}_b,J_b}(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3)\exp[-i\boldsymbol{K}_b\cdot\boldsymbol{R}]=u(r_{21},r_{31}),$$

on account of the charge independence of nuclear forces. A convenient form for $u(r_{21}, r_{31})$, used in variational calculations of E_a, E_b (10), is:

$$(20a)\qquad u(r_{21},r_{31})=(\beta^5/96\pi^3)^{\frac{1}{2}}\frac{\exp[-(\beta/2)\sqrt{(\varrho)^2+(\varrho')^2}]}{\sqrt[4]{(\varrho)^2+(\varrho')^2}},$$

where,

$$(20b)\qquad \varrho=\frac{\sqrt{3}}{2}|\boldsymbol{r}_{21}-\boldsymbol{r}_{31}|,\qquad \varrho'=\frac{1}{2}|\boldsymbol{r}_{21}+\boldsymbol{r}_{31}|,\qquad \boldsymbol{R}=\frac{1}{3}(\boldsymbol{r}_1+\boldsymbol{r}_2+\boldsymbol{r}_3),$$

and the parameter β is related to the mean square radius of the proton density distribution function in ${}^3_2\mathrm{He}$ via (eqs. (15b), (14a), (14b), (19a)–(20b))

$$(20c)\qquad \langle r^2\rangle_a\equiv\int r^2\,\mathcal{D}_a(r)\,\mathrm{d}\boldsymbol{r}=\int r^2\langle u|\delta(\boldsymbol{r}-\boldsymbol{r}_1)|u\rangle\,\mathrm{d}\boldsymbol{r}=\frac{20}{3}\frac{1}{\beta^2},$$

the co-ordinates $\boldsymbol{r}$, $\boldsymbol{r}_1$ being taken, in eq. (20c), relative to the center of mass co-ordinate $\boldsymbol{R}$.

We now have, remembering that the wavefunctions $|M_a\rangle$, $|M_b\rangle$ are spatially spherical, and in a manner similar to that in eq. (15a).

$$(21a)\qquad \langle M_b|\sum_i\tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)|M_a\rangle=\langle M_b|\sum_i\tau_i^{(-)}j_0(\nu r_i)\varphi(r_i)|M_a\rangle\simeq$$
$$\simeq\langle M_b|\sum_i\tau_i^{(-)}j_0(\nu r_i)|M_a\rangle\left(1-\frac{Zm'_\mu}{137}\frac{\langle M_b|\sum_i\tau_i^{(-)}j_0(\nu r_i)r_i|M_a\rangle}{\langle M_b|\sum_i\tau_i^{(-)}j_0(\nu r_i)|M_a\rangle}\right)\simeq$$
$$\simeq\langle M_b|\sum_i\tau_i^{(-)}j_0(\nu r_i)|M_a\rangle\left(1-\frac{Zm'_\mu}{137\nu}\gamma_1(\nu^2\langle r^2\rangle_a)^{\frac{1}{2}}\frac{1-\frac{1}{6}(\gamma_3/\gamma_1)(\nu^2\langle r^2\rangle_a)}{1-\frac{1}{6}(\nu^2\langle r^2\rangle_a)}\right),$$

and, similarly,

$$(21b)\qquad \langle M_b|\sum_i\tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)\boldsymbol{\sigma}_i|M_a\rangle=\langle M_b|\sum_i\tau_i^{(-)}j_0(\nu r_i)\varphi(r_i)\boldsymbol{\sigma}_i|M_a\rangle\simeq$$
$$\simeq\langle M_b|\sum_i\tau_i^{(-)}j_0(\nu r_i)\boldsymbol{\sigma}_i|M_a\rangle\left(1-\frac{Zm'_\mu}{137\nu}\gamma_1(\nu^2\langle r^2\rangle_a)^{\frac{1}{2}}\frac{1-\frac{1}{6}(\gamma_3/\gamma_1)(\nu^2\langle r^2\rangle_a)}{1-\frac{1}{6}(\nu^2\langle r^2\rangle_a)}\right).$$

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(10) J. IRVING: *Phil. Mag.*, **42**, 338 (1951).

Further, using eqs. (19a), (19b),

$$(22a)\quad \langle M_b|\sum_i \tau_i^{(-)} j_0(\nu r_i)|M_a\rangle = \langle M_b|\sum_i \tau_i^{(-)} \exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i)|M_a\rangle =$$

$$= \sum_{i=1}^{3} \langle v_{J_b,M_b}|\tau_i^{(-)}|v_{J_a,M_a}\rangle\langle u \exp[i\boldsymbol{K}_b\cdot\boldsymbol{R}]|\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]|u\exp[i\boldsymbol{K}_a\cdot\boldsymbol{R}]\rangle =$$

$$= -\frac{1}{3}\delta_{M_b,M_a}\int (u(\varrho,\varrho'))^2\left(\exp\left[-\frac{2}{3}i\boldsymbol{\nu}\cdot\boldsymbol{\rho}\right] + \exp\left[-\frac{1}{3}i\boldsymbol{\nu}\cdot\boldsymbol{\rho} - \frac{1}{\sqrt{3}}i\boldsymbol{\nu}\cdot\boldsymbol{\rho}'\right] + \right.$$

$$\left. + \exp\left[-\frac{1}{3}i\boldsymbol{\nu}\cdot\boldsymbol{\rho} + \frac{1}{\sqrt{3}}i\boldsymbol{\nu}\cdot\boldsymbol{\rho}'\right]\right) d\boldsymbol{\rho}\, d\boldsymbol{\rho}' =$$

$$= -\frac{1}{3}\delta_{M_b,M_a}\left(J\left(-\frac{2}{3},0\right) + J\left(-\frac{1}{3},-\frac{1}{\sqrt{3}}\right) + J\left(-\frac{1}{3},\frac{1}{\sqrt{3}}\right)\right),$$

where

$$(22b)\quad J(x,y) \equiv \int (u(\varrho,\varrho'))^2 \exp[-ix\boldsymbol{\nu}\cdot\boldsymbol{\rho}]\exp[-iy\boldsymbol{\nu}\cdot\boldsymbol{\rho}')\, d\boldsymbol{\rho}\, d\boldsymbol{\rho}' =$$

$$= (4\pi)^2\left(\frac{\beta^5}{96\pi^3}\right)\int_0^\infty\int_0^\infty \frac{\exp[-\beta\sqrt{(\varrho)^2+(\varrho')^2}]}{\sqrt{(\varrho)^2+(\varrho')^2}} j_0(x\nu\varrho)j_0(y\nu\varrho')(\varrho)^2(\varrho')^2\, d\varrho\, d\varrho' =$$

$$= \beta^5/[\beta^2 + (x^2+y^2)\nu^2]^{\frac{5}{2}}.$$

Thus from eqs. (22a), (22b), and (20c):

$$(23a)\quad \sum_{M_b,M_a}|\langle M_b|\sum_i \tau_i^{(-)} j_0(\nu r_i)|M_a\rangle|^2 = \frac{2}{(1+(1/15)(\nu^2\langle r^2\rangle_a))^5},$$

so that

$$(23b)\quad \sum_{M_b,M_a}|\langle M_b|\sum_i \tau_i^{(-)}|M_a\rangle|^2 = 2$$

and in a similar manner:

$$(23c)\quad \sum_{M_b,M_a}|\langle M_b|\sum_i \tau_i^{(-)} j_0(\nu r_i)\boldsymbol{\sigma}_i|M_a\rangle|^2 = \frac{6}{(1+(1/15)(\nu^2\langle r^2\rangle_a))^5},$$

whence

$$(23d)\quad \sum_{M_b,M_a}|\langle M_b|\sum_i \tau_i^{(-)}\boldsymbol{\sigma}_i|M_a\rangle|^2 = 6$$

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Eqs. (11a), (21a), (21b), (23a)–(23d), (11c) and use of $(g_A^{(\beta)}/g_V^{(\beta)})^2 = (1.21)^2$ yield:

$$(24)\qquad R({}^3_2\mathrm{He} \rightleftarrows {}^3_1\mathrm{H}) \cong \left(1 + \frac{1}{15}\,(\nu^2\langle r^2\rangle_a)\right)^{-5}\cdot$$

$$\cdot\left(1 - \frac{2Zm'_\mu}{137\nu}\left(\frac{3}{4}\right)^{\frac{1}{2}}(\nu^2\langle r^2\rangle_a)^{\frac{1}{2}}\,\frac{1-\frac{1}{6}\cdot\frac{5}{3}\,(\nu^2\langle r^2\rangle_a)}{1-\frac{1}{6}\,(\nu^2\langle r^2\rangle_a)}\right)\cdot\left[\left(\frac{G_V^{(\mu)}}{g_V^{(\beta)}}\right)^2\frac{1+(\Gamma_A^{(\mu)}/G_V^{(\mu)})^2\cdot 3}{1+(g_A^{(\beta)}/g_V^{(\beta)})^2\cdot 3}\right] \cong$$

$$\cong \left(1 - \frac{1}{3}\,(\nu^2\langle r^2\rangle_a) + \frac{1}{15}\,(\nu^2\langle r^2\rangle_a)^2 + \ldots\right)\cdot$$

$$\cdot\left(1 - \frac{2Zm'_\mu}{137\nu}\left(\frac{3}{4}\right)^{\frac{1}{2}}(\nu^2\langle r^2\rangle_a)^{\frac{1}{2}}\,\frac{1-\frac{1}{6}\cdot\frac{5}{3}\,(\nu^2\langle r^2\rangle_a)}{1-\frac{1}{6}\,(\nu^2\langle r^2\rangle_a)}\right)\cdot(1.07) \equiv K'_1({}^3_2\mathrm{He})\cdot K'_2({}^3_2\mathrm{He})\cdot(1.07)\,,$$

where, in addition, we have used for simplicity in the term $\sim Z/137$, values of γ_1, γ_3 (eqs. (15b), (21a), (21b)) which are appropriate to $\mathcal{D}_a(r) \sim \exp[-r/\alpha]$, rather than $\mathcal{D}_a(r) = \langle u|\delta(\mathbf{r}-\mathbf{r}_1)|u\rangle$ (as given by eqs. (14a), (14b), (19a)–(20b)). This last approximation is quite minor since comparison of eq. (24) with eqs. (18a), (17b) shows that $\gamma_4 = 2.33$, while $\mathcal{D}_a(r) \sim \exp[-r/\alpha]$ predicts $\gamma_4 = 2.50$. The value of β and so of $(\langle r^2\rangle_a)^{\frac{1}{2}}$ (eqs. (20a), (20c)) may be determined by comparison of the observed Coulomb energy of ${}^3_2$He and the Coulomb energy of ${}^3_2$He calculated on the basis of the wavefunction of eq. (20a); the value of $(\langle r^2\rangle_a)^{\frac{1}{2}}$ obtained in this way works out to be [11]:

$$(25)\qquad (\langle r^2\rangle_a)^{\frac{1}{2}} = 1.78\cdot 10^{-13}\ \mathrm{cm}\,.$$

This, together with $\nu = 103.3$ MeV/c gives, using eq. (24),

$$(26)\qquad R({}^3_2\mathrm{He} \rightleftarrows {}^3_1\mathrm{H}) = 0.791$$

so that, from eq. (10), with $f({}^3_1\mathrm{H} \to {}^3_2\mathrm{He}) = 2.94\cdot 10^{-6}$ and consequently $k = 1.04\cdot 10^{12}$, we obtain:

$$(27)\qquad \frac{w^{(\mu)}}{w^{(\beta)}} = 8.24\cdot 10^{11}\,, \qquad ({}^3_2\mathrm{He} \rightleftarrows {}^3_1\mathrm{H}),$$

which, with

$$(28)\qquad w^{(\beta)} = 1.77\cdot 10^{-9}\ \mathrm{s}^{-1}\ ({}^3_1\mathrm{H} \to {}^3_2\mathrm{He})$$

finally yields

$$(29)\qquad w^{(\mu)} = 1.46\cdot 10^{3}\ \mathrm{s}^{-1}\ ({}^3_2\mathrm{He} \to {}^3_1\mathrm{H})\,.$$

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(11) We use eq. (15) of reference (10).

B) ${}^6_3\mathrm{Li} \rightleftarrows {}^6_2\mathrm{He}$. – The transition between the ground states is here: $(J_a=1,\ \mathcal{P}_a=+1) \rightleftarrows (J_a=0,\ \mathcal{P}_b=+1)$. As noted after eq. (13),

$$\langle M_b|\sum_i \tau_i^{(-)} \exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)|M_a\rangle\,,$$

as well as $\langle M_b|\sum_i \tau_i^{(-)}|M_a\rangle$ vanish in this case so that eq. (11*a*) becomes:

$$(30)\qquad R=\left(\frac{\Gamma_A^{(\mu)}}{g_A^{(\beta)}}\right)^2\frac{\int(\mathrm{d}\hat{\boldsymbol{\nu}}/4\pi)\sum_{M_b,M_a}|\langle M_b|\sum_i\tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)\boldsymbol{\sigma}_i|M_a\rangle|^2}{\sum_{M_b,M_a}|\langle M_b|\sum_i\tau_i^{(-)}\boldsymbol{\sigma}_i|M_a\rangle|^2}\,.$$

We adopt *LS*-coupling shell model wavefunctions [12] of the form:

$$\left\{\alpha\psi_a({}^3S_1)+\sqrt{1-|\alpha|^2}\,\psi_a({}^3D_1)\right\},\qquad\qquad \psi_b({}^1S_0)\,,$$

for $|M_a\rangle$, $|M_b\rangle$. Then, since as in eq. (7) there is no interference between emitted neutrinos with different l, and since neutrinos with odd l or with even $l>2$ cannot be emitted in view of the values of J_a, $\mathcal{P}_a$; J_b, $\mathcal{P}_b$, we have

$$(31a)\qquad \int\frac{\mathrm{d}\hat{\boldsymbol{\nu}}}{4\pi}\sum_{M_b,M_a}|\langle M_b|\sum_i\tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)\boldsymbol{\sigma}_i|M_a\rangle|^2=$$
$$=\int\frac{\mathrm{d}\hat{\boldsymbol{\nu}}}{4\pi}\sum_{M_b,M_a}\Big\{|\langle M_b|\sum_i\tau_i^{(-)}j_0(\nu r_i)\varphi(r_i)\boldsymbol{\sigma}_i|M_a\rangle|^2+$$
$$+|\langle M_b|\sum\tau_i^{(-)}j_2(\nu r_i)\sqrt{4\pi\cdot 3}\,Y_{2,0}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{r}}_i)\varphi(r_i)\boldsymbol{\sigma}_i|M_a\rangle|^2\Big\}=$$
$$=\int\frac{\mathrm{d}\hat{\boldsymbol{\nu}}}{4\pi}\sum_{M_b,M_a}\Big\{|\alpha|^2|\langle\psi_b({}^1S_0)|\sum_i\tau_i^{(-)}j_0(\nu r_i)\varphi(r_i)\boldsymbol{\sigma}_i|\psi_a({}^3S_1)\rangle|^2+$$
$$+(1-|\alpha|^2)|\langle\psi_b({}^1S_0)|\sum_i\tau_i^{(-)}j_2(\nu r_i)\sqrt{4\pi\cdot 3}\,Y_{2,0}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{r}}_i)\varphi(r_i)\boldsymbol{\sigma}_i|\psi_a({}^3D_1)\rangle|^2\Big\},$$

and

$$(31b)\qquad \sum_{M_b,M_a}|\langle M_b|\sum_i\tau_i^{(-)}\boldsymbol{\sigma}_i|M_a\rangle|^2=\sum_i|\alpha|^2|\langle\psi_b({}^1S_0)|\sum_i\tau_i^{(-)}\boldsymbol{\sigma}_i|\psi_a({}^3S_1)\rangle|^2\,.$$

Now comparison of calculated and observed ${}^6_3\mathrm{Li}$ magnetic moments and ${}^6_2\mathrm{He}\to{}^6_3\mathrm{Li}$ β decay lifetimes [12], the latter involving eq. (31*b*) for the corresponding nuclear matrix elements, indicates that $|\alpha|^2\simeq 0.88$. The second term in eq. (31*a*) may then be neglected since $(1-|\alpha|^2)\simeq 0.12$ and since, as

(12) E. Feenberg: *Shell Theory of the Nucleus* (Princeton, 1955), ch. 3, 8.

as already noted, the d-wave neutrino nuclear matrix element,

$$|\langle\psi_b(^1S_0)|\sum_i \tau_i^{(-)} j_2(\nu r_i)\sqrt{4\pi\cdot 3}\, Y_{2,0}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{r}}_i)\varphi(r_i)\boldsymbol{\sigma}_i|\psi_a(^3D_1)\rangle|\,,$$

is, at least for light nuclei, in general appreciably smaller than the s-wave neutrino nuclear matrix element,

$$|\langle\psi_b(^1S_0)|\sum_i \tau_i^{(-)} j_0(\nu r_i)\varphi(r_i)\boldsymbol{\sigma}_i|\psi_a(^3S_1)\rangle|\,.$$

In addition, we can write analogously to eq. (21b),

$$\text{(32)}\qquad \langle M_b|\sum_i \tau_i^{(-)} j_0(\nu r_i)\varphi(r_i)\boldsymbol{\sigma}_i|M_a\rangle = \alpha\langle\psi_b(^1S_0)|\sum_i \tau_i^{(-)} j_0(\nu r_i)\varphi(r_i)\boldsymbol{\sigma}_i|\psi_a(^3S_1)\rangle \cong$$

$$\cong \alpha\langle\psi_b(^1S_0)|\sum_i \tau_i^{(-)} j_0(\nu r_i)\boldsymbol{\sigma}_i|\psi_a(^3S_1)\rangle\left(1-\frac{Zm'_\mu}{137\nu}\,\gamma_1\,(\nu^2\langle r^2\rangle_a)^{\frac{1}{2}}\,\frac{1-\frac{1}{6}(\gamma_3/\gamma_1)\,(\nu^2\langle r^2\rangle_a)}{1-\frac{1}{6}(\nu^2\langle r^2\rangle_a)}\right).$$

Then, adopting explicit forms for $\psi_a(^3S_1)$, $\psi_b(^1S_0)$ which consider the four nucleons in the 1s-shell as inert during the ground state $\leftrightarrow$ ground state transition and using a standard notation ([12]):

$$\text{(33}a\text{)}\qquad \psi_a(^3S_1) = u_{E_a\cdot\,\mathcal{D}_a\cdot J_a}(\boldsymbol{r}_1,\boldsymbol{r}_2)\,v_{J_a\cdot M_a}(\sigma_1^{(3)},\sigma_2^{(3)};\tau_1^{(3)},\tau_2^{(3)}) =$$

$$= \Big(\sum_m C_{0,0}^{1,m;1,-m}\, Y_{1,m}(\theta_1,\varphi_1)\, Y_{1,-m}(\theta_2,\varphi_2)\, X(r_1)\, X(r_2)\Big)\cdot\big(^3\chi_{M_a}(\sigma_1^{(3)},\sigma_2^{(3)})\,^1\Phi_0(\tau_1^{(3)},\tau_2^{(3)})\big)\,,$$

$$\text{(33}b\text{)}\qquad \psi_b(^1S_0) = u_{E_b\cdot\,\mathcal{D}_b\cdot J_b}(\boldsymbol{r}_1,\boldsymbol{r}_2)\,v_{J_b\cdot M_b}(\sigma_1^{(3)},\sigma_2^{(3)};\tau_1^{(3)},\tau_2^{(3)}) =$$

$$= \Big(\sum_m C_{0.0}^{1,m1.-m}\, Y_{1.m}(\theta_1,\varphi_1)\, Y_{1.-m}(\theta_2,\varphi_2)\, X(r_1)\, X(r_2)\Big)\cdot\big(^1\chi_{M_b}(\sigma_1^{(3)},\sigma_2^{(3)})\,^3\Phi_{-1}(\tau_1^{(3)},\tau_2^{(3)})\big)\,,$$

with

$$\text{(33}c\text{)}\qquad X(r) = \left(\frac{8}{3\pi^{\frac{1}{2}}d^5}\right)^{\frac{1}{2}}\exp[-r^2/2d^2]\,r\,;\qquad \mathcal{D}_a(r) = \frac{1}{4\pi}(X(r))^2 =$$

$$= \left(\frac{2}{3\pi^{\frac{3}{2}}d^5}\right)\exp[-r^2/d^2]\,r^2\,;\qquad \langle r^2\rangle_a \equiv \int r^2\,\mathcal{D}_a(r)\,\mathrm{d}\boldsymbol{r} = \frac{5}{2}\,d^2\,,$$

we get:

$$\text{(34}a\text{)}\qquad \sum_{M_b\cdot M_a}|\langle\psi_b(^1S_0)|\sum_i \tau_i^{(-)} j_0(\nu r_i)\boldsymbol{\sigma}_i|\psi_a(^3S_1)\rangle|^2 = 6\left(\int_0^\infty j_0(\nu r)(X(r))^2\, r^2\,\mathrm{d}r\right)^2 =$$

$$= 6\left(\exp\left[-\left(\frac{1}{5}\right)(\nu^2\langle r^2\rangle_a)\right]\left[1-\frac{2}{15}(\nu^2\langle r^2\rangle_a)+\frac{1}{225}(\nu^2\langle r^2\rangle_a)^2\right]\right) =$$

$$= 6\left(1-\frac{1}{3}(\nu^2\langle r^2\rangle_a)+\frac{23}{450}(\nu^2\langle r^2\rangle_a)^2+\ldots\right),$$

so that

$$\sum_{M_b, M_a} \left|\langle \psi_b(^1S_0) \left| \sum_i \tau_i^{(-)} \boldsymbol{\sigma}_i \right| \psi_a(^3S_1)\rangle\right|^2 = 6 \; . \tag{34b}$$

The $X(r)$ in eq. (33c) is appropriate to the p-shell radial wavefunction of a shell model oscillator potential [13] and $\mathcal{D}_a(r)$ is the corresponding (directionally-averaged-over) density distribution function of the protons in the $1p$-shell introduced in eqs. (16a), (16b). Thus, eqs. (30), (31a) (with neglect of the term $\sim (1-|\alpha|^2)$), (31b), (32), (34a), (34b) and (11c), yield:

$$R(^6_3\mathrm{Li} \rightleftarrows {}^6_2\mathrm{He}) \cong \left(1 - \frac{1}{3}(\nu^2\langle r^2\rangle_a) + \frac{23}{450}(\nu^2\langle r^2\rangle_a)^2 + \ldots\right) \cdot \tag{36}$$

$$\cdot\left(1 - \frac{2Zm'_\mu}{137\nu}\left(\frac{128}{45\pi}\right)^{\frac{1}{2}}(\nu^2\langle r^2\rangle_a)^{\frac{1}{2}}\,\frac{1-\frac{1}{6}\cdot\frac{6}{5}(\nu^2\langle r^2\rangle_a)}{1-\frac{1}{6}(\nu^2\langle r^2\rangle_a)}\right)\cdot\left(\frac{\Gamma_A^{(\mu)}}{g_A^{(\beta)}}\right)^2 \equiv K'_1(^6_3\mathrm{Li})\cdot K'_2(^6_3\mathrm{Li})\cdot(1.08)\,,$$

the factor $|\alpha|^2$ cancelling out. In eq. (36) we have used values of γ_1, γ_3 (eqs. (15b), (32)) appropriate to $\mathcal{D}_a(r) = (1/4\pi)(X(r))^2 \sim \exp[-r^2/d^2)r^2$ as in eq. (33c). Comparison of eq. (36) with eqs. (18b), (17b) shows this $\mathcal{D}_a(r)$ predicts $\gamma_4 = 1.40$.

The value of d and so of $(\langle r^2\rangle_a)^{\frac{1}{2}}$ (eq. (33c)) may be determined by comparison of the observed and the calculated electron elastic scattering from $^6_3$Li and works out to be [14]:

$$d = 1.52\cdot 10^{-13}\ \text{cm}\,; \qquad (\langle r^2\rangle_a)^{\frac{1}{2}} = 2.40\cdot 10^{-13}\ \text{cm}\,. \tag{37}$$

This, together with $\nu = 100.7$ MeV/c, gives, using eq. (36),

$$R(^6_3\mathrm{Li} \rightleftarrows {}^6_2\mathrm{He}) = 0.619 \tag{38}$$

so that, from eq. (10), with $f(^6_2\mathrm{He} \to {}^6_3\mathrm{Li}) = 1.05\cdot 10^3$ and consequently $k = 10.0\cdot 10^3$, we obtain:

$$\frac{w^{(\mu)}}{w^{(\beta)}} = 2.06\cdot 10^3\,, \quad (^6_3\mathrm{Li} \rightleftarrows {}^6_2\mathrm{He}), \tag{39}$$

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(13) I. Talmi: *Helv. Phys. Acta*, **25**, 185 (1952).

(14) R. Hofstadter: reference (7); D. G. Ravenhall: *Rev. Mod. Phys.*, **30**, 430 (1958); G. R. Burleson and R. Hofstadter: *Phys. Rev.*, **112**, 1282 (1958). Actually, as discussed in detail by Burleson and Hofstadter, the density distribution function of all the protons in $^6_3$Li which best describes the electron elastic scattering corresponds to a $\mathcal{D}_a(r)$ for the $1p$-shell protons which differs somewhat from the form: $\exp[-r^2/d^2]r^2$. As a result our numerical value for $(\langle r^2\rangle_a)^{\frac{1}{2}}$ in eq. (37) is not very accurate.

which, with

$$w^{(\beta)} = 0.868\ \text{s}^{-1}\ ({}^6_2\text{He} \rightarrow {}^6_3\text{Li})\,, \tag{40}$$

finally yields:

$$w^{(\mu)} = 1.79 \cdot 10^3\ \text{s}^{-1}\ ({}^6_3\text{Li} \rightarrow {}^6_2\text{He})\,. \tag{41}$$

C) ${}^{12}_{6}\text{C} \rightleftarrows {}^{12}_{5}\text{B}$. – The transition between the ground states is here:

$$(J_a = 0,\ \mathcal{P}_a = +1) \rightleftarrows (J_b = 1,\ \mathcal{P}_b = +1)\,.$$

As noted after eq. (13), $\langle M_b | \sum_i \tau_i^{(-)} \exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i) | M_a \rangle$ as well as $\langle M_b | \sum_i \tau_i^{(-)} | M_a \rangle$ vanish in this case so that eq. (11*a*) becomes:

$$R = \left(\frac{\Gamma_A^{(\mu)}}{g_A^{(\beta)}}\right)^2 \frac{\int (\mathrm{d}\hat{\boldsymbol{\nu}}/4\pi) \sum_{M_b, M_a} |\langle M_b | \sum_i \tau_i^{(-)} \exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)\boldsymbol{\sigma}_i | M_a \rangle|^2}{\sum_{M_b, M_a} |\langle M_b | \sum_i \tau_i^{(-)} \boldsymbol{\sigma}_i | M_a \rangle|^2}\,. \tag{42}$$

We adopt *jj*-coupling shell model wave functions ([12]) for $|M_a\rangle$ and $|M_b\rangle$. Assuming again that the four nucleons in the 1*s*-shell are inert during the ground state $\leftrightarrow$ ground state transition one may write:

$$|M_a\rangle = \alpha_a \psi_{a,\alpha} + \beta_a \psi_{a,\beta}\,, \tag{43a}$$

$$|M_b\rangle = \alpha_b \psi_{b,\alpha} + \beta_b \psi_{b,\beta}\,, \tag{43b}$$

where

$$\psi_{a,\alpha} \equiv \psi\big((1p_{\frac{3}{2}})^4;\ (1p_{\frac{3}{2}})^4\big)\,; \qquad \psi_{a,\beta} \equiv \equiv \psi\big((1p_{\frac{3}{2}})^3(1p_{\frac{1}{2}})^1;\ (1p_{\frac{3}{2}})^3(1p_{\frac{1}{2}})^1\big)\,; \qquad |\alpha_a|^2 + |\beta_a|^2 = 1\,, \tag{43c}$$

$$\psi_{b,\alpha} \equiv \psi\big((1p_{\frac{3}{2}})^3;\ (1p_{\frac{3}{2}})^4(1p_{\frac{1}{2}})^1\big)\,; \qquad \psi_{b,\beta} \equiv \equiv \psi\big((1p_{\frac{3}{2}})^2(1p_{\frac{1}{2}})^1;\ (1p_{\frac{3}{2}})^4(1p_{\frac{1}{2}})^1\big)\,; \qquad |\alpha_b|^2 + |\beta_b|^2 = 1\,, \tag{43d}$$

and where the contribution of all configurations other than those indicated is considered small. Since as in eq. (7) or in eq. (31*a*), there is no interference between emitted neutrinos with different l, and since neutrinos with odd l or with even $l > 2$ cannot be emitted in view of the values of J_a, $\mathcal{P}_a$; J_b, $\mathcal{P}_b$,

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we have:

$$(44a)\quad \int \frac{d\hat{\nu}}{4\pi} \sum_{M_b, M_a} |\langle M_b \sum_i \tau_i^{(-)} \exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)\boldsymbol{\sigma}_i | M_a\rangle|^2 =$$

$$= \int \frac{d\hat{\nu}}{4\pi} \sum_{M_b, M_a} \{ |\langle M_b | \sum_i \tau_i^{(-)} j_0(\nu r_i)\varphi(r_i)\boldsymbol{\sigma}_i | M_a\rangle|^2 +$$

$$+ |\langle M_b | \sum_i \tau_i^{(-)} j_2(\nu r_i)\sqrt{4\pi\cdot 3}\, Y_{2,0}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{r}}_i)\varphi(r_i)\boldsymbol{\sigma}_i | M_a\rangle|^2\} =$$

$$= \int \frac{d\hat{\nu}}{4\pi} \sum_{M_b, M_a} \{ |\alpha_b^*\alpha_a\langle\psi_{b,\alpha} | \sum_i \tau_i^{(-)} j_0(\nu r_i)\varphi(r_i)\boldsymbol{\sigma}_i | \psi_{a,\alpha}\rangle +$$

$$+ \alpha_b^*\beta_a\langle\psi_{b,\alpha} | \sum_i \tau_i^{(-)} j_0(\nu r_i)\varphi(r_i)\boldsymbol{\sigma}_i | \psi_{a,\beta}\rangle + \beta_b^*\beta_a\langle\psi_{b,\beta} | \sum_i \tau_i^{(-)} j_0(\nu r_i)\varphi(r_i)\boldsymbol{\sigma}_i | \psi_{a,\beta}\rangle|^2 +$$

$$+ |\alpha_b^*\alpha_a\langle\psi_{b,\alpha} | \sum_i \tau_i^{(-)} j_2(\nu r_i)\sqrt{4\pi\cdot 3}\, Y_{2,0}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{r}}_i)\varphi(r_i)\boldsymbol{\sigma}_i | \psi_{a,\alpha}\rangle +$$

$$+ \alpha_b^*\beta_a\langle\psi_{b,\alpha} | \sum_i \tau_i^{(-)} j_2(\nu r_i)\sqrt{4\pi\cdot 3}\, Y_{2,0}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{r}}_i)\varphi(r_i)\boldsymbol{\sigma}_i | \psi_{a,\beta}\rangle +$$

$$+ \beta_b^*\beta_a\langle\psi_{b,\beta} | \sum_i \tau_i^{(-)} j_2(\nu r_i)\sqrt{4\pi\cdot 3}\, Y_{2,0}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{r}}_i)\varphi(r_i)\boldsymbol{\sigma}_i | \psi_{a,\beta}\rangle|^2\},$$

and,

$$(44b)\quad \sum_{M_b, M_a} |\langle M_b | \sum_i \tau_i^{(-)}\boldsymbol{\sigma}_i | M_a\rangle|^2 = \sum_{M_b, M_a} |\alpha_b^*\alpha_a\langle\psi_{b,\alpha} | \sum_i \tau_i^{(-)}\boldsymbol{\sigma}_i | \psi_{a,\alpha}\rangle +$$

$$+ \alpha_b^*\beta_a\langle\psi_{b,\alpha} | \sum_i \tau_i^{(-)}\boldsymbol{\sigma}_i | \psi_{a,\beta}\rangle + \beta_b^*\beta_a\langle\psi_{b,\beta} | \sum_i \tau^{(-)}\boldsymbol{\sigma}_i | \psi_{a,\beta}\rangle|^2,$$

the matrix elements

$$\langle\psi_{b,\beta} | \sum_i \tau_i^{(-)} j_0(\nu r_i)\varphi(r_i)\boldsymbol{\sigma}_i | \psi_{a,\alpha}\rangle, \qquad \langle\psi_{b,\beta} | \sum_i \tau_i^{(-)} j_2(\nu r_i)\sqrt{4\pi\cdot 3}\, Y_{2,0}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{r}}_i)\varphi(r_i)\boldsymbol{\sigma}_i | \psi_{a,\alpha}\rangle,$$

$$\langle\psi_{b,\beta} | \sum_i \tau_i^{(-)}\boldsymbol{\sigma}_i | \psi_{a,\alpha}\rangle,$$

all vanishing since the configurations associated with $\psi_{b,\beta}$ and $\psi_{a,\alpha}$ differ from each other by more than a single orbital.

We have further:

$$(45a)\quad \frac{\langle\psi_{b,\alpha} | \sum_i \tau_i^{(-)} j_0(\nu r_i)\varphi(r_i)\boldsymbol{\sigma}_i | \psi_{a,\alpha}\rangle}{f(b, \alpha, M_b; a, \alpha, M_a)} = \frac{\langle\psi_{b,\alpha} | \sum_i \tau_i^{(-)} j_0(\nu r_i)\varphi(r_i)\boldsymbol{\sigma}_i | \psi_{a,\beta}\rangle}{f(b, \alpha, M_b; a, \beta, M_a)} =$$

$$= \frac{\langle\psi_{b,\beta} | \sum_i \tau_i^{(-)} j_0(\nu r_i)\varphi(r_i)\boldsymbol{\sigma}_i | \psi_{a,\beta}\rangle}{f(b, \beta, M_b; a, \beta, M_a)} = \int_0^\infty j_0(\nu r)\varphi(r)(X(r))^2 r^2\, dr,$$

$$(45b)\quad \frac{\langle\psi_{b,\alpha}|\sum_i \tau_i^{(-)} j_2(\nu r_i)\sqrt{4\pi\cdot 3}\, Y_{2,0}(\boldsymbol{\nu}\cdot\hat{\boldsymbol{r}}_i)\varphi(r_i)\boldsymbol{\sigma}_i|\psi_{a,\alpha}\rangle}{\boldsymbol{g}(b,\alpha,M_b;a,\alpha,M_a)} =$$

$$= \frac{\langle\psi_{b,\alpha}|\sum_i \tau_i^{(-)} j_2(\nu r_i)\sqrt{4\pi\cdot 3}\, Y_{2,0}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{r}}_i)\varphi(r_i)\boldsymbol{\sigma}_i|\psi_{a,\beta}\rangle}{\boldsymbol{g}(b,\alpha,M_b;a,\beta,M_a)} =$$

$$= \frac{\langle\psi_{b,\beta}|\sum_i \tau_i^{(-)} j_2(\nu r_i)\sqrt{4\pi\cdot 3}\, Y_{2,0}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{r}}_i)\varphi(r_i)\boldsymbol{\sigma}_i|\psi_{a,\beta}\rangle}{\boldsymbol{g}(b,\beta,M_b;a,\beta,M_a)} = \int_0^\infty j_2(\nu r)\varphi(r)(X(r))^2 r^2\,\mathrm{d}r\,,$$

$$(45c)\quad \frac{\langle\psi_{b,\alpha}|\sum_i \tau_i^{(-)}\boldsymbol{\sigma}_i|\psi_{a,\alpha}\rangle}{\boldsymbol{f}(b,\alpha,M_b;a,\alpha,M_a)} = \frac{\langle\psi_{b,\alpha}|\sum_i \tau_i^{(-)}\boldsymbol{\sigma}_i|\psi_{a,\beta}\rangle}{\boldsymbol{f}(b,\alpha,M_b;a,\beta,M_a)} =$$

$$= \frac{\langle\psi_{b,\beta}|\sum_i \tau_i^{(-)}\boldsymbol{\sigma}_i|\psi_{a,\beta}\rangle}{\boldsymbol{f}(b,\beta,M_b;a,\beta,M_a)} = \int_0^\infty (X(r))^2 r^2\,\mathrm{d}r = 1\,,$$

where $X(r)$ is the $1p$-shell ($1p_{\frac{3}{2}}$ or $1p_{\frac{1}{2}}$) radial wavefunction given in eq. (33c) and $\boldsymbol{f}(b,\alpha,M_b;a,\alpha,M_a)$, ..., $\boldsymbol{g}(b,\beta,M_b;a,\beta,M_a)$ are vector numerical coefficients with values depending on the magnetic quantum numbers M_b, M_a and on the « configuration indices » b, α; a, β; etc. Thus, from eqs. (44a)–(45c),

$$(46a)\quad \int\frac{\mathrm{d}\hat{\boldsymbol{\nu}}}{4\pi}\sum_{M_b,M_a}|\langle M_b|\sum_i \tau_i^{(-)}\exp[-i\boldsymbol{\nu}\cdot\boldsymbol{r}_i]\varphi(r_i)\boldsymbol{\sigma}_i|M_a\rangle|^2 =$$

$$= \int\frac{\mathrm{d}\hat{\boldsymbol{\nu}}}{4\pi}\sum_{M_b,M_a}\Bigg\{\Bigg(\int_0^\infty j_0(\nu r)\varphi(r)(X(r))^2 r^2\mathrm{d}r\Bigg)^2|\alpha_b^*\alpha_a\boldsymbol{f}(b,a,M_b;a,\alpha,M_a) +$$

$$+\alpha_b^*\beta_a\boldsymbol{f}(b,\alpha,M_b;a,\beta,M_a)+\beta_b^*\beta_a\boldsymbol{f}(b,\beta,M_b;a,\beta,M_a)|^2 +$$

$$+\Bigg(\int_0^\infty j_2(\nu r)\varphi(r)(X(r))^2 r^2\,\mathrm{d}r\Bigg)^2|\alpha_b^*\alpha_a\boldsymbol{g}(b,\alpha,M_b;a,\alpha,M_a)+\alpha_b^*\beta_a\boldsymbol{g}(b,\alpha,M_b;a,\beta,M_a) +$$

$$+\beta_b^*\beta_a\boldsymbol{g}(b,\beta,M_b;a,\beta,M_a)|^2\Bigg\} \equiv \Bigg(\int_0^\infty j_0(\nu r)\varphi(r)(X(r))^2 r^2\,\mathrm{d}r\Bigg)^2\eta_0 +$$

$$+\Bigg(\int_0^\infty j_2(\nu r)\varphi(r)\,(X(r))^2\,r^2\mathrm{d}r\Bigg)^2\eta_2\,,$$

and,

$$(46b)\quad \sum_{M_b,M_a}|\langle M_b|\sum_i \tau_i^{(-)}\boldsymbol{\sigma}_i|M_a\rangle^2 = \sum_{M_b,M_a}|\alpha_b^*\alpha_a\,\boldsymbol{f}(b,\alpha,M_b;a,\alpha,M_a) +$$

$$+\alpha_b^*\beta_a\,\boldsymbol{f}(b,\alpha,M_b;a,\beta,M_a)+\beta_b^*\beta_a\,\boldsymbol{f}(b,\beta,M_b;a,\beta,M_a)|^2$$

$$\equiv \eta_0\,,$$

while by an argument similar to that in eqs. (15a) or (21a), (21b), or (32), we have,

$$(47a)\quad \int_0^\infty j_0(\nu r)\varphi(r)(X(r))^2 r^2\,\mathrm{d}r \cong \int_0^\infty j_0(\nu r)(X(r))^2 r^2\,\mathrm{d}r\cdot \cdot\left(1-\frac{Zm'_\mu}{137\nu}\gamma_1(\nu^2\langle r^2\rangle_a)^{\frac12}\frac{1-\frac16(\gamma_3/\gamma_1)(\nu^2\langle r^2\rangle_a)}{1-\frac16(\nu^2\langle r^2\rangle_a)}\right),$$

$$(47b)\quad \int_0^\infty j_2(\nu r)\varphi(r)(X(r))^2 r^2\,\mathrm{d}r \cong \int_0^\infty j_2(\nu r)(X(r))^2 r^2\,\mathrm{d}r\cdot\left(1-\frac{Zm'_\mu}{137\nu}\gamma_3(\nu^2\langle r^2\rangle_a)^{\frac12}\right)\cong$$

$$\cong \int_0^\infty j_2(\nu r)(X(r))^2 r^2\,\mathrm{d}r\cdot\left(1-\frac{Zm'_\mu}{137\nu}\gamma_1(\nu^2\langle r^2\rangle_a)^{\frac12}\frac{1-\frac16(\gamma_3/\gamma_1)\nu^2\langle r^2\rangle_a)}{1-\frac16(\nu^2\langle r^2\rangle_a)}\right).$$

Eqs. (42), (46a)–(47b), yield:

$$(48)\quad R\cong\left[\left(\int_0^\infty j_0(\nu r)(X(r))^2 r^2\,\mathrm{d}r\right)^2+\frac{\eta_2}{\eta_0}\left(\int_0^\infty j_2(\nu r)(X(r))^2 r^2\,\mathrm{d}r\right)^2\right]\cdot \cdot\left[1-\frac{2Zm'_\mu}{137\nu}\left(\frac{128}{45\pi}\right)^{\frac12}(\nu^2\langle r^2\rangle_a)^{\frac12}\frac{1-\frac16\cdot\frac65(\nu^2\langle r^2\rangle_a)}{1-\frac16(\nu^2\langle r^2\rangle_a)}\right]\cdot\left(\frac{\Gamma_A^{(\mu)}}{g_A^{(\beta)}}\right)^2=$$

$$=\left[\exp\left(-\frac15(\nu^2\langle r^2\rangle_a)\right)\left(1-\frac{2}{15}(\nu^2\langle r^2\rangle_a)+\frac{1}{225}(\nu^2\langle r^2\rangle_a)^2\right)+\right.$$

$$\left.+\frac{\eta_2}{\eta_0}\left(\exp\left(-\frac15(\nu^2\langle r^2\rangle_a)\right)\frac{1}{225}(\nu^2\langle r^2\rangle_a)^2\right)\right]\cdot$$

$$\cdot\left[1-\frac{2Zm'_\mu}{137\nu}\left(\frac{128}{45\pi}\right)^{\frac12}(\nu^2\langle r^2\rangle_a)^{\frac12}\frac{1-\frac16\cdot\frac65(\nu^2\langle r^2\rangle_a)}{1-\frac16(\nu^2\langle r^2\rangle_a)}\right]\cdot\left(\frac{\Gamma_A^{(\mu)}}{g_A^{(\beta)}}\right)^2=$$

$$=\left[\left(1-\frac13(\nu^2\langle r^2\rangle_a)+\frac{23}{450}(\nu^2\langle r^2\rangle_a)^2+\dots\right)+\frac{\eta_2}{\eta_0}\left(\frac{1}{225}(\nu^2\langle r^2\rangle_a)^2+\dots\right)\right]\cdot$$

$$\cdot\left[1-\frac{2Zm'_\mu}{137\nu}\left(\frac{128}{45\pi}\right)^{\frac12}(\nu^2\langle r^2\rangle_a)^{\frac12}\frac{1-\frac16\cdot\frac65(\nu^2\langle r^2\rangle_a)}{1-\frac16(\nu^2\langle r^2\rangle_a)}\right]\cdot\left(\frac{\Gamma_A^{(\mu)}}{g_A^{(\beta)}}\right)^2,$$

where we have used values of the integrals, or equivalently values of $\gamma_4\equiv\langle r^4\rangle_a/(\langle r^2\rangle_a)^2$ etc., and values of γ_1,γ_3, appropriate to $\mathcal{D}_a(r)=(1/4\pi)(X(r))^2$ being $\sim\exp[-r^2/d^2]r^2$ (eq. (33c)), *i.e.* being the (directionally-averaged-over) density distribution of the protons in the 1p-shell. Eq. (48) demonstrates the

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relative unimportance of d-wave neutrino emission since

$$\left(\int_0^\infty j_2(\nu r)(X(r))^2 r^2\,\mathrm{d}r\right)^2 \Big/ \left(\int_0^\infty j_0(\nu r)(X(r))^2 r^2\,\mathrm{d}r\right)^2 =$$

$$= \frac{(1/225)(\nu^2\langle r^2\rangle_a)^2}{1-(2/15)(\nu^2\langle r^2\rangle_a)+(1/225)(\nu^2\langle r^2\rangle_a)^2} \simeq 0.01\,;$$

as a consequence η_2/η_0 need not be known accurately and we estimate it in the approximation of jj-coupling without configuration mixing, *i.e.* in the approximation $\alpha_a \approx 1$, $\alpha_b \approx 1$ so that $|M_a\rangle \approx \psi_{a,\alpha}$, $|M_b\rangle \approx \psi_{b,\alpha}$. Under these circumstances we have, remembering eqs. (45a)–(46b):

$$\text{(49)}\qquad \frac{\eta_2}{\eta_0} \approx \frac{\sum\limits_{M_b, M_a} |\boldsymbol{g}(b, \alpha, M_b;\, a, \alpha, M_a)|^2}{\sum\limits_{M_b, M_a} |\boldsymbol{f}(b, \alpha, M_b;\, a, \alpha, M_a)|^2} =$$

$$= \frac{\sum\limits_{M_b, M_a} |\langle \psi_{b,\alpha}| \sum\limits_i \tau_i^{(-)} j_2(\nu r_i)\sqrt{4\pi\cdot 3}\, Y_{2,0}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{r}}_i)\varphi(r_i)\boldsymbol{\sigma}_i |\psi_{a,\alpha}\rangle|^2 \Big/ \left(\int_0^\infty j_2(\nu r)\varphi(r)(X(r))^2 r^2\,\mathrm{d}r\right)^2}{\sum\limits_{M_b, M_a} |\langle \psi_{b,\alpha}| \sum\limits_i \tau_i^{(-)} j_0(\nu r_i)\varphi(r_i)\boldsymbol{\sigma}_i |\psi_{a,\alpha}\rangle|^2 \Big/ \left(\int_0^\infty j_0(\nu r)\varphi(r)(X(r))^2 r^2\,\mathrm{d}r\right)^2}\,,$$

which can be evaluated using, in a standard notation ([12]):

$$\text{(50}a\text{)}\qquad \psi_{a,\alpha} = \frac{1}{\sqrt{2}} \sum_m C^{\frac{3}{2},m;\frac{3}{2},-m}_{0,0}\,\cdot$$

$$\cdot\big[W_{\frac{3}{2},m;\frac{1}{2},\frac{1}{2}}(\boldsymbol{r}_2, \ldots, \boldsymbol{r}_8;\, \sigma_2^{(3)}, \ldots, \sigma_8^{(3)};\, \tau_2^{(3)}, \ldots, \tau_8^{(3)})\,\chi_{\frac{3}{2},-m}(\boldsymbol{r}_1, \sigma_1^{(3)})\varphi_{\frac{1}{2},-\frac{1}{2}}(\tau_1^{(3)}) -$$

$$- W_{\frac{3}{2},m;\frac{1}{2},-\frac{1}{2}}(\boldsymbol{r}_2, \ldots, \boldsymbol{r}_8;\, \sigma_2^{(3)}, \ldots, \sigma_8^{(3)};\, \tau_2^{(3)}, \ldots, \tau_8^{(3)})\,\chi_{\frac{3}{2},-m}(\boldsymbol{r}_1, \sigma_1^{(3)})\varphi_{\frac{1}{2},\frac{1}{2}}(\tau_1^{(3)})\big]\,,$$

$$\text{(50}b\text{)}\qquad \psi_{b,\alpha} = \frac{1}{\sqrt{8}} \sum_\nu \mathcal{P}_\nu \sum_m C^{\frac{3}{2},m;\frac{1}{2},M_b-m}_{1,M_b}\,\cdot$$

$$\cdot\big[W_{\frac{3}{2},m;\frac{1}{2},-\frac{1}{2}}(\boldsymbol{r}_2, \ldots, \boldsymbol{r}_8;\, \sigma_2^{(3)}, \ldots, \sigma_8^{(3)};\, \tau_2^{(3)}, \ldots, \tau_8^{(3)})\,\chi_{\frac{1}{2},M_b-m}(\boldsymbol{r}_1, \sigma_1^{(3)})\varphi_{\frac{1}{2},-\frac{1}{2}}(\tau_1^{(3)})\big]\,,$$

where $W_{\frac{3}{2},m;\frac{1}{2},\pm\frac{1}{2}}$, $\chi_{\frac{3}{2},-m}$, $\chi_{\frac{1}{2},M_b-m}$, $\varphi_{\frac{1}{2},\pm\frac{1}{2}}$, $\mathcal{P}_\nu$, are, respectively, a) completely antisymmetrized wavefunctions of seven nucleons with total angular momentum $\frac{3}{2}$, total isobaric-spin $\frac{1}{2}$, and corresponding third components m, $\pm\frac{1}{2}$, arising from the configurations $(1p_{\frac{3}{2}})^4(1p_{\frac{3}{2}})^3$, $(1p_{\frac{3}{2}})^3(1p_{\frac{3}{2}})^4$; b) single particle $1p_{\frac{3}{2}}$-shell space-spin wavefunctions with third component of angular momentum $=-m$; c) single particle $1p_{\frac{1}{2}}$-shell space-spin wavefunctions with third component of angular momentum $= M_b - \frac{1}{2}$; d) isobaric-spin $\frac{1}{2}$ wavefunctions with third component of isobaric-spin $=\pm\frac{1}{2}$; e) permutation operator interchanging the values of $\boldsymbol{r}_1$, $\sigma_1^{(3)}$, $\tau_1^{(3)}$, with those of $\boldsymbol{r}_i$, $\sigma_i^{(3)}$, $\tau_i^{(3)}$ $(i = 2, \ldots, 8)$.

Eqs. (49), (50a), (50b) yield:

$$(51)\qquad \frac{\eta_2}{\eta_0} \approx$$

$$\approx \frac{\dfrac{\sum_{M_b, M_a} \left| 2 \sum_m C_{1,M_b}^{\frac{3}{2},m;\frac{1}{2},M_b-m} C_{0,0}^{\frac{3}{2},m;\frac{3}{2},-m} \cdot \sum_{\sigma^{(3)}} \int \chi^*_{\frac{1}{2},M_b-m}(\boldsymbol{r}, \sigma^{(3)}) \left(j_2(\nu r)\sqrt{4\pi \cdot 3}\, Y_{2,0}(\hat{\boldsymbol{\nu}}\cdot\hat{\boldsymbol{r}})\varphi(r)\boldsymbol{\sigma}\right)\chi_{\frac{3}{2},-m}(\boldsymbol{r}, \sigma^{(3)})\, \mathrm{d}\boldsymbol{r} \right|^2}{\left(\int_0^\infty j_2(\nu r)\varphi(r)(X(r))^2 r^2 \,\mathrm{d}r\right)^2}}{\dfrac{\sum_{M_b, M_a} \left| 2 \sum_m C_{1,M_b}^{\frac{3}{2},m;\frac{1}{2},M_b-m} C_{0,0}^{\frac{3}{2},m;\frac{3}{2},-m} \cdot \sum_{\sigma^{(3)}} \int \chi^*_{\frac{1}{2},M_b-m}(\boldsymbol{r}, \sigma^{(3)}) \left(j_0(\nu r)\varphi(r)\boldsymbol{\sigma}\right) \chi_{\frac{3}{2},-m}(\boldsymbol{r}, \sigma^{(3)})\, \mathrm{d}\boldsymbol{r} \right|^2}{\left(\int_0^\infty j_0(\nu r)\varphi(r)(X(r))^2 r^2 \,\mathrm{d}r\right)^2}}$$

$$= \frac{2/3}{16/3} = \frac{1}{8}.$$

It may also be mentioned that comparison of calculated and observed ${}^{12}_{5}\mathrm{B} \to {}^{12}_{6}\mathrm{C}$ β decay lifetimes ([12]), the latter involving eq. (46b) for the corresponding nuclear matrix elements, indicates that $\eta_0 \simeq 1$ while eq. (51) shows that the assumption of jj-coupling without configuration mixing, *i.e.* the assumption of $\alpha_a = 1$, $\alpha_b = 1$, gives $\eta_0 = 16/3$. On the other hand, the value of the ratio η_2/η_0 is probably far more correctly rendered by the $\alpha_a = 1$, $\alpha_b = 1$ approximation than the value of η_0 or η_2 separately; in any case and as we have already noted, the value of R in eq. (48) is quite insensitive to the value of η_2/η_0. Thus we have, from eqs. (48), (51), 11c):

$$(52)\qquad R({}^{12}_{6}\mathrm{C} \rightleftarrows {}^{12}_{5}\mathrm{B}) \cong \left(\left(1 - \frac{1}{3}(\nu^2\langle r^2\rangle_a) + \frac{23}{450}(\nu^2\langle r^2\rangle_a)^2 + \ldots\right) + \right.$$
$$\left. + \frac{1}{8}\left(\frac{1}{225}(\nu^2\langle r^2\rangle_a)^2 + \ldots\right)\right) \cdot \left(1 - \frac{2Zm'_\mu}{137\nu}\left(\frac{128}{45\pi}\right)^{\frac{1}{2}} (\nu^2\langle r^2\rangle_a)^{\frac{1}{2}} \frac{1 - \frac{1}{6}\cdot\frac{6}{5}(\nu^2\langle r^2\rangle_a)}{1 - \frac{1}{6}(\nu^2\langle r^2\rangle_a)}\right) \cdot$$
$$\cdot \left(\frac{\Gamma_A^{(\mu)}}{g_A^{(\beta)}}\right)^2 \equiv K'_1({}^{12}_{6}\mathrm{C}) \cdot K'_2({}^{12}_{6}\mathrm{C}) \cdot (1.08),$$

which may be compared with eqs. (18c), (17b), (17c).

The value of d and so of $(\langle r^2\rangle_a)^{\frac{1}{2}}$ (eq. (33c)) may be determined by comparison of the observed and the calculated electron elastic scattering from ${}^{12}_{6}\mathrm{C}$ and works out to be ([15]):

$$(53)\qquad d = 1.59 \cdot 10^{-13}\ \mathrm{cm}\ ; \qquad (\langle r^2\rangle_a)^{\frac{1}{2}} = 2.52 \cdot 10^{-13}\ \mathrm{cm}.$$

([15]) R. Hofstadter: reference ([7]); D. G. Ravenhall: reference ([14]); H. F. Ehrenberg, R. Hofstadter, U. Meyer-Berkhout, D. R. Ravenhall and S. E. Sobottka: *Phys. Rev.*, **113**, 666 (1959).

This together with $\nu = 91.4$ MeV/c, gives, using eq. (52),

$$R({}^{12}_{6}\mathrm{C} \rightleftarrows {}^{12}_{5}\mathrm{B}) = 0.612 \tag{54}$$

so that, from eq. (10), with $f({}^{12}_{5}\mathrm{B} \to {}^{12}_{6}\mathrm{C}) = 5.625 \cdot 10^5$ and consequently $k = 1.29 \cdot 10^2$, we obtain:

$$\frac{w^{(\mu)}}{w^{(\beta)}} = 2.37 \cdot 10^2\,, \qquad ({}^{12}_{6}\mathrm{C} \rightleftarrows {}^{12}_{5}\mathrm{B}), \tag{55}$$

which, with

$$w^{(\beta)} = 33.15\ \mathrm{s}^{-1}\ ({}^{12}_{5}\mathrm{B} \to {}^{12}_{6}\mathrm{C}) \tag{56}$$

finally yields:

$$w^{(\mu)} = 7.86 \cdot 10^3\ \mathrm{s}^{-1}\ ({}^{12}_{6}\mathrm{C} \to {}^{12}_{5}\mathrm{B})\,. \tag{57}$$

5. – Discussion.

We collect our theoretical value of the ground state $\leftrightarrow$ ground state muon capture transition rates (eqs. (29), (41), (57)):

$$\left\{\begin{aligned} & w^{(\mu)}({}^{3}_{2}\mathrm{He} \to {}^{3}_{1}\mathrm{H}) = 1.46 \cdot 10^3\ \mathrm{s}^{-1}\,, \\ & w^{(\mu)}({}^{6}_{3}\mathrm{Li} \to {}^{6}_{2}\mathrm{He}) = 1.79 \cdot 10^3\ \mathrm{s}^{-1}\,, \\ & w^{(\mu)}({}^{12}_{6}\mathrm{C} \to {}^{12}_{5}\mathrm{B}) = 7.86 \cdot 10^3\ \mathrm{s}^{-1}\,. \end{aligned}\right. \tag{58}$$

These values of $w^{(\mu)}$ are subject to an uncertainty in the corresponding nuclear matrix elements arising largely from the uncertainty in the determination of the mean square radius of the density distribution of the capturing protons, $\langle r^2 \rangle_a$ (cf. eqs. (24), (25); (36), (37), (52), (53))—it is perhaps not too optimistic to estimate the resultant uncertainty in our $w^{(\mu)}$ as $(10 \div 15)\,\%$. Also, a theoretical calculation due to WOLFENSTEIN, based on assumptions essentially similar to ours, yields $w^{(\mu)}({}^{12}_{6}\mathrm{C} \to {}^{12}_{5}\mathrm{B}) = 7.4 \cdot 10^3\ \mathrm{s}^{-1}$ [16]. As regards experimental va-

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[16] L. WOLFENSTEIN: paper G.5 at Conference on Weak Interactions, Gatlinburg, Tenn. (1958).

lues of $w^{(\mu)}$, data are at present available only in the ${}^{12}_{6}C \to {}^{12}_{5}B$ case and are:

$$(59)\quad [w^{(\mu)}({}^{12}_{6}C \to {}^{12}_{6}B)]_{\text{expt.}} = \begin{cases} (9.05 \pm 0.95)\cdot 10^3\ s^{-1}\ (^{17}), \\ \left(10.2\cdot\dfrac{9}{10} \pm 0.5\right)\cdot 10^3\ s^{-1} = (9.18 \pm 0.5)\cdot 10^3\ s^{-1}\ (^{18}), \\ \left(7.3\cdot\dfrac{9}{10} \pm 1.1\right)\cdot 10^3\ s^{-1} = (6.6\ \pm 1.1)\cdot 10^3\ s^{-1}\ (^{19}), \\ \left(7.6\cdot\dfrac{9}{10} \pm 1.5\right)\cdot 10^3\ s^{-1} = (6.8\ \pm 1.5)\cdot 10^3\ s^{-1}\ (^{20}), \\ \left(6.5\cdot\dfrac{9}{10} \pm 1.5\right)\cdot 10^3\ s^{-1} = (5.9\ \pm 1.5)\cdot 10^3\ s^{-1}\ (^{2}). \end{cases}$$

It is thus clear, from eqs. (58) and (59), that the theoretical value of $w^{(\mu)}({}^{12}_{6}C = {}^{12}_{5}B)$ agrees, within the overall theoretical and experimental uncertainties, with the corresponding experimental value. In this way one finds support for the combination of basic assumptions adopted in the present work, *viz.*:

a) « Universality » between muon-bare nucleon and electron-bare nucleon coupling constants which implies, for the effective muon-dressed nucleon and electron-dressed nucleon coupling constants:

$$\frac{g_A^{(\mu)}}{g_A^{(\beta)}} = 0.999\,; \qquad \frac{g_V^{(\mu)}}{g_V^{(\beta)}} = 0.972\,, \qquad \text{(eq. (3}f\text{))}.$$

b) The presence of an « induced » pseudoscalar interaction ([3,4]) with an effective muon-dressed nucleon coupling constant $g_P^{(\mu)} = 8g_A^{(\beta)}$ (eq. (3*f*)).

c) The presence of anomalous nucleon magnetic moment contributions in the effective muon-dressed nucleon interaction associated with the concept of a « conserved vector current » ([5]) (eqs. (2*c*), (4), (5*c*), (9*b*), (11*b*), (11*c*)). In particular, if the assumption of the « conserved vector current » is abandoned, and the anomalous nucleon magnetic moment contributions to the effective muon-dressed nucleon coupling constants omitted from eqs. (4), (5*c*),

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([17]) F. B. Harrison, H. V. Argo, H. W. Kruse and A. D. McGuire: paper S.4 at Conference on Weak Interactions, Gatlinburg, Tenn. (1958). It is shown in this paper that about 9 out of every 10 muon captures to bound states of ${}^{12}_{5}B$ go to the ground state of ${}^{12}_{5}B$ and we have corrected the results of references ([18-20,2]) accordingly.

([18]) J. O. Burgman, J. Fischer, B. Leontic, A. Lundby, R. Meunier, J. P. Stroot and J. D. Teja: *Phys. Rev. Lett.*, **1**, 469 (1958).

([19]) J. G. Fetkovich, T. H. Fields and R. L. McIlwain: paper G.7 at Conference on Weak Interactions, Gatlinburg, Tenn. (1958).

([20]) W. Love, S. Marder, I. Nadelhaft, R. Siegel and A. E. Taylor: paper G.8 at Conference on Weak Interactions, Gatlinburg, Tenn. (1958); R. Siegel: private communication.

(9b), (11b), (11c), the quantity $(\Gamma_A^{(\mu)}/g_A^{(\beta)})^2$ of eq. (11c) is 0.87 rather than 1.08 and the $R({}^{12}_{6}\text{C} \to {}^{12}_{5}\text{B})$ of eqs. (52), (54), which is proportional to $(\Gamma_A^{(\mu)}/g_A^{(\beta)})^2$, becomes $0.612\cdot(0.87/1.08) = 0.494$. The corresponding $w^{(\mu)}({}^{12}_{6}\text{C} \to {}^{12}_{5}\text{B})$ is, from eqs. (10), (54)–(57), equal to $(7.86\cdot 10^3\ \text{s}^{-1})\cdot(0.87/1.08) = 6.34\cdot 10^3\ \text{s}^{-1}$ which appears to disagree with the first pair of experimental values in eq. (59). If therefore future and more accurate experiments and calculations uphold, a) this first pair of experimental values and, b) our theoretical estimate of $\{R({}^{12}_{6}\text{C} \to {}^{12}_{5}\text{B})/(\Gamma_A^{(\mu)}/g_A^{(\beta)})^2\} = 0.612/1.08 = 0.567$ (eqs. (52)–(54)), one will possess quite suggestive evidence from the field of muon capture for the validity of the concept of a « conserved vector current ».

In conclusion it may be worthwhile mentioning that the total muon capture transition rate for ${}^{12}_{6}\text{C} \to {}^{12}_{5}\text{B}$ is, both from experiment ([21]) and from a closure type theoretical calculation ([22]), equal to $44\cdot 10^3\ \text{s}^{-1}$; thus using eq. (58), about 18% of the muon captures in ${}^{12}_{6}\text{C}$ result in the formation of ${}^{12}_{5}\text{B}$ in its ground state. No experiments as yet exist even for the total muon capture transition rates in ${}^{3}_{2}\text{H} \to {}^{3}_{1}\text{H}$ and ${}^{6}_{3}\text{Li} \to {}^{6}_{2}\text{He}$; however a closure type theoretical calculation ([22]) for the ${}^{3}_{2}\text{He} \to {}^{3}_{1}\text{H}$ case predicts a total muon capture transition rate of $2.5\cdot 10^3\ \text{s}^{-1}$, so that using eq. (58), about 58% of the muon captures in ${}^{3}_{2}\text{He}$ are expected to form ${}^{3}_{1}\text{H}$ in its ground state.

([21]) J. C. Sens: reference ([1]); J. C. Sens, R. A. Swanson, V. L. Telegdi and D. D. Yovanovitch: reference ([1]); T. H. Fields, R. L. McIlwain and J. G. Fetkovich: paper G.6 at Conference on Weak Interactions, Gatlinburg, Tenn. (1958).

([22]) H. Primakoff: reference ([1]).

RIASSUNTO (*)

Si calcolano le percentuali delle reazioni per cattura di muoni: $\mu^- + {}^{3}_{2}\text{He} \to {}^{3}_{1}\text{H} + \nu$, $\mu^- + {}^{6}_{3}\text{Li} \to {}^{6}_{2}\text{He} + \nu$, $\mu^- + {}^{12}_{6}\text{C} \to {}^{12}_{5}\text{B} + \nu$, coi nuclei che ne derivano prodotti nei loro stati fondamentali assumendo l'« universalità » tra le costanti d'accoppiamento muone-nucleone nudo ed elettrone-nucleone nudo. L'interazione pseudoscalare « indotta » e i termini addizionali derivanti dall'assunzione di una « corrente vettoriale conservata » sono inclusi nell'Hamiltoniana effettiva muone-nucleone. I rapporti degli elementi della materia nucleare per le catture dei muoni e pei corrispondenti decadimenti β si stimano dapprima in modo approssimato « indipendentemente da modelli » usando opportune funzioni di distribuzione della densità protonica nucleare e poi si calcolano più dettagliatamente sulla base di funzioni d'onda variazionali di prova per ${}^{3}_{2}\text{He}$, ${}^{3}_{1}\text{H}$, e funzioni d'onda d'accoppiamento LS e jj del modello a shell (con mistura di configurazioni) per ${}^{6}_{3}\text{Li}$, ${}^{6}_{2}\text{He}$ e ${}^{12}_{6}\text{C}$, ${}^{12}_{5}\text{B}$ rispettivamente. La percentuale di cattura calcolata per $\pi^- + {}^{12}_{6}\text{C} \to {}^{12}_{5}\text{B} + \nu$ si accorda con l'esperienza; mancano, tuttavia le esperienze per gli altri due casi.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XII, N. 4 16 Maggio 1959

Theory of the Cosmic Ray Equator (*).

R. Ingraham

Department of Physics, New Mexico State University - State College, N. Mex.

(ricevuto l'11 Gennaio 1959)

Summary. — The 45° westward shift of the cosmic ray equator from the geomagnetic equator is explained by assuming that there exist ionized atmospheric layers which rotate more slowly than the earth. The resultant magnetic field beyond the layers is then a dipole field whose magnetic north pole is west of the familiar geomagnetic north pole. The condition that this longitudinal shift be 45° determines $\Delta\omega \equiv \omega' - \omega$ as a function of certain radial moments of the conductivity of the layer (ω', ω = rotational velocities of layer and earth, respectively). The effect would vanish if either $\Delta\omega = 0$ or if the earth's intrinsic dipole axis were not tilted.

1. – Introduction.

Measurements of the intensity of cosmic ray secondaries near the equator have shown (1) that the « cosmic ray equator » does not coincide with the geomagnetic equator, but is shifted about 45° to the west. Subsequent measurements in higher latitudes (2) complicate this simple picture somewhat. The charged primaries, coming to us roughly isotropically, are effective probes revealing the structure of the magnetic field in the earth's neighborhood; their secondaries, emanating from showers produced in the upper atmosphere, thus report faithfully the state of the geomagnetic field above a certain altitude.

We find that the existence of a horizontal atmospheric shear is sufficient to account for this shift of the cosmic ray equator from the geomagnetic equator.

(*) This work was supported by the Office of Naval Research.

(1) *IGY Bulletin*, no. 15 (September 1958), p. 16.

(2) According to verbal remarks made by Dr. J. Winckler and Dr. J. Chamberlain.

That is, we hypothesize that the atmosphere does not rotate rigidly with the earth, but its rate of rotation drops off with increasing altitude. Then as a result of the interaction of the *intrinsic* geomagnetic field (*i.e.*, that originating inside the earth from core currents, etc.) with rotating ionized atmospheric layers, the magnetic field seen by the incoming primaries external to these layers differs both from the intrinsic dipole field and from the conventional geomagnetic dipole field. The latter is defined as that dipole field which coincides with the geomagnetic field as measured at the ground; its dipole has magnetic moment $\mu \simeq 8.06 \cdot 10^{25}$ cgs emu and its axis MN (³) is tilted at the angle $\alpha = 11°$ from the rotation (geographical) axis N (³). (The geomagnetic dipole field is not the same as the intrinsic dipole field on the present theory, since the geomagnetic field at the earth's surface contains an atmospheric contribution of the same order as that which makes the external field differ from the intrinsic dipole field outside the ionized layers.) This external field (to first order, *i.e.*, to terms linear in the conductivity) is itself exactly a dipole field. Its magnetic north pole $(MN)_{ext}$ turns out to be west of MN. Its magnetic moment and tilt also differ slightly from those of the geomagnetic dipole by amounts depending on certain atmospheric parameters.

The current system in the ionized layer or layers responsible for this modification of the intrinsic dipole field are mainly of two kinds: 1) the dynamo current caused by rotation of the ionized atmosphere with some angular velocity ω' across the (tilted) intrinsic dipole field, and 2) the « Ohm's law current » driven by the electric field associated with the time varying part (of frequency $\omega \equiv$ rotational frequency of the earth) of the intrinsic dipole field. These two current densities are of the same order and define first order of smallness for current systems.

The exact calculation reveals two surprising features not obtainable from an order of magnitude calculation. The effect is proportional to $\Delta\omega \sin\gamma$, where $\Delta\omega = \omega' - \omega$ and γ is the tilt of the intrinsic dipole from N. Thus 1) the effect vanishes if $\sin\gamma \to 0$, *i.e.*, the contribution of the large « untilted » part (proportional to $\cos\gamma$) of the intrinsic dipole field vanishes exactly although the associated dynamo current system is substantial. Also 2) there is the « accidental » feature that ω' and ω tend to cancel so that there is also no effect for the case of no atmospheric shear. This is surprising in view of the fact that the two current systems mentioned above have quite different form and do not at all cancel when $\omega' = \omega$. Finally, a correct westward rotation of the dipole axis results from the natural assumption $\Delta\omega < 0$.

(³) MN and N shall primarily mean the magnetic north pole and geographic north pole, respectively, in this paper. Now and then they will also denote the axes themselves, as here, when no confusion can arise. Similarly for $(MN)_{intr}$, $(MN)_{ext}$, etc.

In Sect. **2** the exact external magnetic field to first order is deduced; and the relation between the total conductivity and mean distance of the layer and the relative shear $\Delta\omega/\omega$ that is implied by the IGY experimental results is given. Sect. **3** discusses several possibilities of a more general treatment which may be expected to contribute a more complicated latitude dependence to this external field. In the Appendix the proof of the formulas (2.8) to (2.10) giving the magnetic potential at any point inside or outside the ionized layer directly in terms of certain radial moment of the current density is carried through.

2. – The external dipole field.

Let the ionized layer be idealized to a spherical shell $A \leqslant r \leqslant B$ of conductivity $\varkappa(r)$, an arbitrary function of r [4] vanishing outside of (A, B). The total current density we are concerned with is given the generalized Ohm's Law

$$\boldsymbol{j}_{\text{total}} = \varkappa\left[\boldsymbol{E}_0 + \frac{1}{c}(\boldsymbol{\omega}' \times \boldsymbol{r}) \times \boldsymbol{B}_0\right] + \boldsymbol{j}_{\text{ret}}\,. \tag{2.1}$$

We write all equations in cgs Heaviside-Lorentz units. Here the second term is the dynamo current. For simplicity we take $\omega' = \text{const} < \omega$ in (A, B); $\boldsymbol{\omega}'$ is parallel to $\boldsymbol{\omega}$. $\boldsymbol{B}_0$ is the earth's *intrinsic* dipole field of which, it turns out, only the « tilted » part

$$\boldsymbol{B}_0' = -\sin\gamma\,\frac{M}{r^3}\,(-2\sin\theta\sin\varphi_r,\ \cos\theta\sin\varphi_r,\ \cos\varphi_r)\,, \tag{2.2}$$

contributes a current with a resultant magnetic field [5]. Here $M > 0$ is proportional to the intrinsic dipole magnetic moment and γ is the tilt of the intrinsic dipole magnetic axis, $(MN)_{\text{intr}}$ say, from N. The coordinates θ and φ_r are ordinary colatitude and longitude except that the prime meridian $\varphi_r = 0$ is taken for convenience to be the meridian through an axis perpendicular to both $(MN)_{\text{intr}}$ and N at geocenter (see Fig. 1). The order of the polar components of a vector

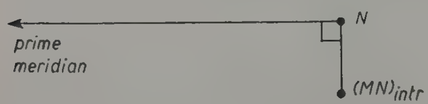

Fig. 1. – View from above the north pole, showing the direction of the prime meridian $\varphi_r = 0$.

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[4] The refinements of allowing an angular and/or time dependence of $\varkappa$ are briefly considered in Sect. **3**.

[5] This important result will not be proved here.

$\boldsymbol{A}$ will always be $(A_r, A_\theta, A_\varphi)$. The first term of (2.1) is the conventional Ohm's law current, where $\boldsymbol{E}_0$ is the solenoidal electric field associated with the time varying part $\boldsymbol{B}_0'$ of $\boldsymbol{B}_0$, or

$$\boldsymbol{E}_0 = -\sin\gamma \frac{M}{r^2}\frac{\omega}{c}(0, \sin\varphi_r, \cos\theta\cos\varphi_r)\,. \tag{2.3}$$

The last term in (2.1) is the «return» current necessary to guarantee charge conservation, or $\nabla\cdot\boldsymbol{j}_{\text{total}} \simeq 0$ in this quasi-steady case. Note $\varkappa\boldsymbol{E}_0$ satisfies $\nabla\cdot\boldsymbol{j} = 0$ by itself. The return current is irrotational and thus contributes nothing to the magnetic field [6]. The current (2.1) is the total first order current, all terms being of order $\varkappa(r\omega/c)(M/r^3)$ (assuming $\omega' \simeq \omega$). There is of course no zero order current.

The magnetic potential can now be written down immediately once we know the components of $\boldsymbol{j}$ in some one fixed set of inertial axes. We have found that the most convenient set of components for this purpose are the circular cartesian components j_σ ($\sigma = +, -, 0$) of the *complex* current density defined by

$$j_+ \equiv j_x + ij_y\,, \qquad j_- \equiv j_x - ij_y\,, \qquad j_0 \equiv j_z\,. \tag{2.4}$$

Here j_x, j_y, and j_z are the complex quantities formed by replacing $\cos\varphi_r$ by $\exp[i\varphi_r]$, etc., in Eqs. (2.2) and (2.3). The reason for using complex quantities is that the analysis is much simplified. The convention for extracting the values of intrinsically real physical quantities from the corresponding complex quantities will always be to take the real part throughout this article. We now need to assume only that the j_σ be a sum of terms of the form

$$(j_l^m)_\sigma = (f_l^m)_\sigma Y_l^{m+\sigma}(\theta, \varphi)\,, \tag{2.5}$$

which is the case in general and is true here in particular. Here θ and φ are inertial polar coordinates with north pole N, hence θ = colatitude and $\varphi \equiv \varphi_r + \omega t$ is essentially local solar time measured in degrees. The $(f_l^m)_\sigma$ are in general r

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([6]) The generalized Ohm's Law (2.1) is that for a steady state, cfr. SPITZER ([7]), Eq. (2.22), and is appropriate here where everything varies with the low frequency ω. Then $\boldsymbol{j}_{\text{ret}}$ includes the effect of a quasi-static electric potential due to a charge distribution inside the layer, cfr. the related discussion in CHAPMAN ([8]), p. 769. This $\boldsymbol{j}_{\text{ret}}$ also involves certain pressure gradients. *E.g.*, in the case of a fully ionized electrically neutral layer, this is simply the ion pressure gradient, the total pressure balancing the Lorentz force within the layer, cfr. SPITZER: loc. cit., Eqs. (2.21), (2.22).

([7]) L. SPITZER: *Physics of Fully Ionized Gases* (New York, 1956).

([8]) S. CHAPMAN and J. BARTELS: *Geomagnetism* (Oxford, 1940).

and t dependent. The Y_L^M are spherical harmonics as used by ROSE, CONDON and SHORTLEY, *et al* ([9]). N.B. especially their sign convention! By linearity, one can treat each harmonic (l, m) separately, then simply add the final results. Now we find that the « tilted » dynamo current (that is, the dynamo current formed with $\boldsymbol{B}_0'$ alone) is the sum of two harmonics $(l, m)=(1, 1)$ and $(3, 1)$, whereas the current $\varkappa \boldsymbol{E}_0$ has only the one harmonic $(l, m)=(1, 1)$. Writing $\boldsymbol{j} \equiv \boldsymbol{j}_{\text{total}}(\boldsymbol{B}_0 \to \boldsymbol{B}_0') - \boldsymbol{j}_{\text{ret}}$, we get

$$(2.6)\qquad \begin{cases} (j_1^1)_+ = 0\,, \\ (j_1^1)_- = iM\dfrac{\varkappa}{c}\sin\gamma\left(4\sqrt{\dfrac{\pi}{3}}\,\omega - \dfrac{12}{5}\sqrt{\dfrac{\pi}{3}}\,\omega'\right)\dfrac{1}{r^2}\,Y_1^0\,, \\ (j_1^1)_0 = iM\dfrac{\varkappa}{c}\sin\gamma\left(2\sqrt{\dfrac{2\pi}{3}}\,\omega - \dfrac{14}{5}\sqrt{\dfrac{2\pi}{3}}\,\omega'\right)\dfrac{1}{r^2}\,Y_1^1\,, \end{cases}$$

and

$$(2.7)\qquad \begin{cases} (j_3^1)_+ = -\,iM\dfrac{\varkappa}{c}\sin\gamma\;4\sqrt{\dfrac{6\pi}{35}}\,\omega'\dfrac{1}{r^2}\,Y_3^2\,, \\ (j_3^1)_- = \quad iM\dfrac{\varkappa}{c}\sin\gamma\,\dfrac{12}{5}\sqrt{\dfrac{\pi}{7}}\;\omega'\dfrac{1}{r^2}\,Y_3^0\,, \\ (j_3^1)_0 = \quad iM\dfrac{\varkappa}{c}\sin\gamma\,\dfrac{8}{5}\sqrt{\dfrac{3\pi}{7}}\,\omega'\dfrac{1}{r^2}\,Y_3^1\,. \end{cases}$$

The arguments of all the spherical harmonics are θ and φ_r. From these formulas, the amplitudes $(f_l^m)_\sigma$ of Eq. (2.5) can be read off immediately. We note that their r and t dependence is contained in the factor $\exp[-i\omega t]\,\varkappa(r)/r^2$.

With the given current (2.5) in the ionized shell (A, B), the solution of Maxwell's equations gives for the magnetic potential U at points internal and external to the shell ([10])

$$(2.8)\qquad (U^{\text{int}})_l^m = -\frac{2\pi i}{l(2l+1)}\Big[\sqrt{(l-m)(l+m+1)}\,(R_l^m)_+ + \\ +\sqrt{(l+m)(l-m+1)}\,(R_l^m)_- + 2m(R_l^m)_0\Big]r^l\,Y_l^m(\theta,\varphi)\,,\qquad (l>0;\ r\leqslant A)\,,$$

$$(2.9)\qquad (U^{\text{ext}})_l^m = \frac{2\pi i}{(l+1)(2l+1)}\Big[\sqrt{(l-m)(l+m+1)}\,(S_l^m)_+ + \\ +\sqrt{(l+m)(l-m+1)}\,(S_l^m)_- + 2m(S_l^m)_0\Big]r^{-(l+1)}\,Y_l^m(\theta,\varphi)\,,\qquad (r\geqslant B)\,,$$

([9]) See say M. ROSE: *Multipole Fields* (London, 1955), formula (1.18).
([10]) The proofs of the following formulas are given in the appendix.

where

$$(R_l^m)_\sigma \equiv \frac{1}{4\pi c}\int_A^B \mathrm{d}r\, r^{1-l}(f_l^m)_\sigma\,, \quad (S_l^m)_\sigma \equiv \frac{1}{4\pi c}\int_A^B \mathrm{d}r\, r^{l+2}(f_l^m)_\sigma\,. \tag{2.10}$$

We mention for completeness that for $l=0$, r^l/l is to be replaced by $-\log(a/r)$ in (2.8), where this U is then normalized to vanish at $r=a$. It is a fact that the real parts of (2.8) and (2.9) are the real magnetic potentials.

Now substituting the $(f_l^m)_\sigma$ given by (2.6) and (2.7), one finds that the (3, 1) harmonic gives no magnetic field, for the contents of the brackets in (2.8) and (2.9) vanish! Therefore external and internal first order fields have only the one harmonic (1, 1). One gets

$$\begin{cases} U_1^{\text{ext}} = \dfrac{8\pi}{3}\sqrt{\dfrac{2\pi}{3}}\,\Delta\omega \sin\gamma\, M_m^{(+1)}\dfrac{M}{r^2}\,Y_1^1(\theta,\varphi_r)\,, \\[2ex] U_1^{\text{int}} = -\dfrac{16\pi}{3}\sqrt{\dfrac{2\pi}{3}}\,\Delta\omega \sin\gamma\, M_m^{(-2)} M r\, Y_1^1(\theta,\varphi_r)\,. \end{cases} \tag{2.11}$$

Here $M_m^{(n)} = \int_A^B \mathrm{d}r\, r^n \varkappa_m(r)$ is the nth radial moment of the conductivity in emu ($\varkappa_m = \varkappa/4\pi c^2$), and $\Delta\omega = \omega' - \omega$. Note the remarkable feature that the field depends only on the difference of the angular velocities ω' and ω! Adding in the zero order intrinsic dipole potential, we get the total magnetic potential $U = U_{\text{intr}} + U_1$ inside and outside the shell

$$U^{\text{int}} = -\frac{M}{r^2}\Big[\cos\gamma\cos\theta - \sin\gamma\sin\theta\sin\varphi_r - \\ -\frac{8\pi}{3}\,\Delta\omega\sin\gamma\, M_m^{(-2)} r^3 \sin\theta\cos\varphi_r\Big]\,, \qquad (r\leqslant A), \tag{2.12}$$

$$U^{\text{ext}} = -\frac{M}{r^2}\Big[\cos\gamma\cos\theta - \sin\gamma\sin\theta\sin\varphi_r + \\ +\frac{4\pi}{3}\,\Delta\omega\sin\gamma\, M_m^{(+1)} \sin\theta\cos\varphi_r\Big]\,. \qquad (r\geqslant B)\,. \tag{2.13}$$

These are real magnetic potentials of course; the real parts of (2.11) have been taken.

The problem is to find the external field in terms of known quantities. To do this we must also find the intrinsic dipole field in passing. Now this latter can be determined from (2.12) in the following way. According to (2.12)

the geomagnetic field we measure at the ground is not the intrinsic dipole field alone, but this field plus an atmospheric contribution. So we postulate that at $r = a$, U^{int} must coincide with the observed geomagnetic dipole field:

$$U^{\text{int}}\big|_{r=a} = -\frac{M_{\text{obs}}}{a^2}\left[\cos\alpha\cos\theta - \sin\alpha\sin\theta\sin(\varphi_r - \varepsilon)\right]. \tag{2.14}$$

Here $M_{\text{obs}} > 0$ is proportional to the observed geomagnetic dipole moment $\mu = 8.06\cdot 10^{25}$ cgs emu; α is the tilt of the observed geomagnetic axis MN, $\alpha \simeq 11°$; and the physical meaning of the phase ε will be clarified in a moment. Then expanding the $\sin(\varphi_r - \varepsilon)$ and comparing with (2.12) at $r = a$, we get the equations

$$\begin{cases} M\cos\gamma = M_{\text{obs}}\cos\alpha\,, \qquad M\sin\gamma = M_{\text{obs}}\sin\alpha\cos\varepsilon\,, \\ M\dfrac{8\pi}{3}\Delta\omega\sin\gamma\, M_m^{(-2)}a^3 = -M_{\text{obs}}\sin\alpha\sin\varepsilon\,. \end{cases} \tag{2.15}$$

These have the solutions for the intrinsic quantities

$$\begin{cases} \operatorname{tg}\varepsilon = -\dfrac{8\pi}{3}\Delta\omega\, M_m^{(-2)}a^3\,, \\ \operatorname{tg}\gamma = \cos\varepsilon\operatorname{tg}\alpha\,, \\ M = \dfrac{\cos\alpha}{\cos\gamma}M_{\text{obs}}\,. \end{cases} \tag{2.16}$$

Substituting these values into (2.13), we obtain

$$U^{\text{ext}} = -\frac{M_{\text{obs}}}{r^2}\Big[\cos\alpha\cos\theta - \sin\alpha\cos\varepsilon\sin\theta\sin\varphi_r + \frac{4\pi}{3}\Delta\omega\, M_m^{(+1)}\sin\alpha\cos\varepsilon\sin\theta\cos\varphi_r\Big]. \tag{2.17}$$

But now this is exactly a dipole field; let us write it as

$$U^{\text{ext}} \equiv -\frac{N}{r^2}\left[\cos\beta\cos\theta - \sin\beta\sin\theta\sin(\varphi_r - \delta)\right]. \tag{2.18}$$

Equating this to (2.17), we get three equations similar to (2.15), with the solutions

$$\operatorname{tg}\delta = \frac{4\pi}{3}\Delta\omega\, M_m^{(+1)}, \quad \operatorname{tg}\beta = \frac{\cos\varepsilon}{\cos\delta}\operatorname{tg}\alpha\,, \quad N = \frac{\cos\alpha}{\cos\beta}M_{\text{obs}}\,. \tag{2.19}$$

This completes the formal solution of the problem.

Now for any dipole field, for example the geomagnetic dipole field (2.14), the colatitude and longitude of the north pole MN is $(\alpha,\ 90°+\varepsilon)$ as we have chosen our coordinates. (To see this, note that $(B_\varphi)_{\max}$ occurs at longitude $180°+\varepsilon$, but this must be 90° east of the meridian through MN). Since $(MN)_{\text{intr}}$ has longitude 90°, MN is thus east or west of $(MN)_{\text{intr}}$ according as $\varepsilon>0$ or $\varepsilon<0$. Therefore by (2.16) and (2.19) we see that MN and $(MN)_{\text{ext}}$ are respectively east and west of $(MN)_{\text{intr}}$ *because* $\Delta\omega<0$. Consequently $(MN)_{\text{ext}}$ is west of MN, as the IGY observations call for. Figure 2 summarizes the relative positions of these various magnetic north poles. Furthermore, the longitudinal difference $\eta=\varepsilon+|\delta|$ of MN and $(MN)_{\text{ext}}$ must be about 45°, so

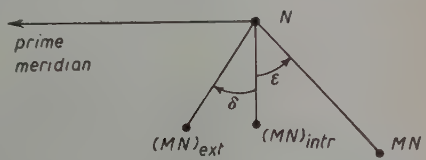

Fig. 2. – Showing the relative longitudinal positions of MN, $(MN)_{\text{intr}}$, and $(MN)_{\text{ext}}$.

$$1=\operatorname{tg}\eta=\frac{\operatorname{tg}\varepsilon+\operatorname{tg}|\delta|}{1-\operatorname{tg}\varepsilon\operatorname{tg}|\delta|}\,. \tag{2.20}$$

Substituting from (2.16) and (2.19), we get

$$\frac{32\pi^2}{9}\,M_m^{(+1)}M_m^{(-2)}a^3|\Delta\omega|^2+\frac{4\pi}{3}\,(M_m^{(+1)}+2M_m^{(-2)}a^3)\,|\Delta\omega|-1=0\,. \tag{2.21}$$

This equation fixes the magnitude $|\Delta\omega|$ of the shear as a function of the radial moments of the conductivity. For the sake of further discussion let us limit ourselves to the case of a *thin shell* at distance $\bar r$. Then $M_m^{(+1)}\simeq\bar rK_m$, $M^{(-2)}a^3\simeq\simeq(a/\bar r)^3\bar rK_m$ in terms of the total conductivity $K_m\equiv\int_A^B\mathrm{d}r\,\varkappa_m(r)$ of the shell, and (2.21) becomes

$$2\left(\frac{a}{\bar r}\right)^3\Theta^2\left|\frac{\Delta\omega}{\omega}\right|^2+\Theta\left\{1+2\left(\frac{a}{\bar r}\right)^3\right\}\left|\frac{\Delta\omega}{\omega}\right|-1=0\,, \tag{2.22}$$

where $\Theta=(4\pi/3)\omega\bar rK_m$. This has the solution [11] for the relative shear

$$\left|\frac{\Delta\omega}{\omega}\right|=\frac{1}{4}\left(\frac{\bar r}{a}\right)^3\left[\left\{1+12\left(\frac{a}{\bar r}\right)^3+4\left(\frac{a}{\bar r}\right)^6\right\}^{\frac12}-\left\{1+2\left(\frac{a}{\bar r}\right)^3\right\}\right]\frac{1}{\Theta}\equiv g\left(\frac{a}{\bar r}\right)\frac{1}{\Theta}\,. \tag{2.23}$$

[11] The negative root must of course be discarded.

The meaning of the important dimensionless ratio Θ is that $|\Delta\omega/\omega| \to 1/\Theta$ as $\bar{r} \to \infty$. From (2.19) one has $\operatorname{tg}\delta = -|\Delta\omega/\omega|\Theta = -g(a/\bar{r})$, hence the longitude of $(MN)_{\text{ext}}$ depends only on the ratio $a/\bar{r}$!

One can now solve for the other parameters of the external dipole field. Always in the thin shell case, the tilt β, of the external dipole axis is given by

$$(2.24) \qquad \operatorname{tg}\beta = \frac{\cos\varepsilon}{\cos\delta}\operatorname{tg}\alpha = \left[\frac{1+g^2}{1+4(a/\bar{r})^6 g^2}\right]^{\frac{1}{2}}\operatorname{tg}\alpha\,.$$

Finally, the external dipole moment, call it ν, is related to the observed geomagnetic dipole moment μ via (2.19), or

$$(2.25) \qquad \nu = \frac{\cos\alpha}{\cos\beta}\mu = \left[\frac{1+g^2\cos^2\alpha\{4(a/\bar{r})^6+\operatorname{tg}^2\alpha\}}{1+4(a/\bar{r})^6 g^2}\right]^{\frac{1}{2}}\mu\,.$$

In the limit $a/\bar{r} \to 0$, $g \to 1$, hence

$$(2.26) \qquad \operatorname{tg}\delta \to -1\,, \quad \operatorname{tg}\beta \to \sqrt{2}\operatorname{tg}\alpha\,, \quad \nu \to (1+\sin^2\alpha)^{\frac{1}{2}}\mu = 1.02\mu\,, \qquad \left(\frac{a}{\bar{r}} \to 0\right).$$

Hence $(MN)_{\text{ext}}$ moves into a position 45° west of $(MN)_{\text{intr}}$ (and MN therefore moves into coincidence with $(MN)_{\text{intr}}$). The other limit $a/\bar{r} = 1$ is also interesting: then $g = (1/4)(\sqrt{17}-3) = .282$. Hence

$$(2.27) \qquad \operatorname{tg}\delta = -.282\,, \quad \operatorname{tg}\beta = .905\operatorname{tg}\alpha\,, \quad \nu = .998\mu\,, \qquad \left(\frac{a}{\bar{r}} = 1\right).$$

In this case $(MN)_{\text{ext}}$ is about 15° 45′ west of $(MN)_{\text{intr}}$.

In applying Eq. (2.23), we are hampered by our ignorance of the location, extent, and conductivity of the ionized layer or layers in question. However, the consideration of possible limiting cases may help to fix ideas as to how great the relative shear should be. At the one extreme, if this effect were produced by the ionosphere alone, the following values could be used: $\bar{r} \simeq a = 6\,370$ km, $K_m = 10^{-7}$ cgs emu. Then using $\omega = 7.29\cdot 10^{-5}\ \text{s}^{-1}$ we would get $g \simeq g(1) = .282$, $\Theta = 2\cdot 10^{-2}$, and hence $|\Delta\omega/\omega| = 1\cdot 10^{1}$. We interpret this non-sensical result as meaning that the ionosphere alone cannot be responsible for so large an effect. On the other hand, if the effect were produced by a rotating band of highly conducting plasma of the general dimensions of the « current rings » used to explain magnetic storms and the aurora ([12]), then

([12]) See J. Chamberlain: *Theories of the Aurora*, chap. 5, in *Adv. in Geophys.* vol. **4** (New York, 1958).

values like $\bar{r}=5a$, $K_m=10^{-4}$ cgs emu (or even higher) might be appropriate. The $g\simeq g(0)=1$ (since g is actually a function of $(a/\bar{r})^3$, this shell is already effectively at infinity) and $\Theta=1\cdot 10^2$. Thus $|\Delta\omega/\omega|=1\cdot 10^{-2}$. Of course the thin shell approximation is not very good for a current ring.

3. – Possible generalizations.

This theory can be generalized in several directions, of which the following seem to us the most immediate. Any of these probably leads to corrections to the dipole U^{ext} by the addition of harmonics higher than the (1, 1) harmonic. This more complicated latitude dependence might be helpful in explaining the features observed in higher latitudes [2].

1) *Higher orders in the conductivity.* The iterative process here is first to solve for the n-th order fields $\boldsymbol{E}_n$, $\boldsymbol{B}_n$ *at points within the ionized shell*, then to form $\boldsymbol{j}_{n+1}$ from the right side of (2.1) with $\boldsymbol{E}_0, \boldsymbol{B}_0 \to \boldsymbol{E}_n, \boldsymbol{B}_n$. From a knowledge of the circular components $(j_{n+1})_\sigma$, the U^{ext}_{n+1} and U^{int}_{n+1} can then immediately be written down by formulas (2.8) to (2.10). One expects that higher harmonics will arise in second order already because the (3, 1) harmonic of $\boldsymbol{j}_2$ will probably *not* lead to a vanishing (3, 1) harmonic in U_2. The « accidental » result $(U_1)^1_3\equiv 0$ was the result of the special r-dependence of $\boldsymbol{B}_0$ ($\propto 1/r^3$). But the r-dependence of $\boldsymbol{B}_1$ *within the ionized layer* will certainly be less simple.

2) *Angle and time dependent conductivity.* If the assumption $\varkappa$=function of r only were relaxed to allow angular and/or time dependence [13], one might certainly expect extra harmonics to appear in U^{ext}_1. The effect of a latitude dependence in the angular velocity ω' of the rotating shell would be similar.

Note added in proof.

After our paper was submitted, we discovered the similar theory of A. BEISER (*Nuovo Cimento*, **8**. 160 (1958)). The guiding idea of his theory and ours is the same, namely that the action of the geomagnetic field on the ionized layer induces currents, which in turn modify the total geomagnetic field outside the ionized layer seen by incoming particles. Our main criticisms of his theory are on the ground that he has neglected several effects equally as important as the one he calculates, thus our numbers will be more accurate than his. We make the following points

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[13] The sketch of a theory of angular and time dependent conductivity can be found in CHAPMAN: loc. cit., p. 771 ff.

1) Beiser considers only the action of the « geoelectric » field (2.3); thus he neglects the dynamo current entirely. But as we have shown, in the case that the ionized layer rotates rigidly with the earth ($\Delta\omega = 0$), the magnetic effects of the dynamo current and geoelectric field-driven current cancel exactly both outside and inside the ionized layer. Thus the result is extremely sensitive to the size of the dynamo current, which must be reckoned in.

2) He neglects the « inverse rotation » of the magnetic north pole caused by the contribution of the ionized layer at points *inside* the layer. *I.e.*, he identifies MN and $(MN)_{\text{intr}}$ (see Fig. 2). Thus he finds for $\operatorname{tg}\eta$ ($\eta =$ « rotation of the cosmic ray equator » $\equiv$ $\equiv$ the longitudinal angle between MN and $(MN)_{\text{ext}}$) a result roughly $\frac{1}{3}$ of what it should be. This can be seen from our Eq. (2.21). In Beiser's theory $\omega' = 0$, therefore $|\Delta\omega| = \omega$, also in his approximation the quadratic term is neglected. Now the neglect of this « inverse rotation » means that he also neglects the term $2M_m^{(-2)}a^3$, therefore gets finally the equation

$$\frac{4\pi}{3}\, M_m^{(+1)}\omega = 1$$

considered as an equation to be solved for $\bar{r}$. But for a thin shell $M_m^{(-2)}a \simeq M_m^{(+1)}$, therefore the left member of this equation should be multiplied by 3.

3) Other criticisms are minor. *E.g.*, Beiser does not calculate the other parameters of the resultant « external » dipole field. Its « tilt » β and magnetic moment ν are calculated in our Eqs. (2.24) and (2.25) respectively.

Appendix

With the current density (2.5), assumed to satisfy $\nabla\cdot\boldsymbol{j} = 0$, the circular cartesian components of the vector potential are

$$(A_l^m)_\sigma = \frac{1}{4\pi c}\int_{\text{shell}} \frac{(j_l^m)'_\sigma}{R}\,\mathrm{d}^3x' , \tag{A.1}$$

where $R \equiv |\boldsymbol{r} - \boldsymbol{r}'|$. Here we neglect radiation, permissible for these very slow motions. We shall treat here only the « external » case $r \geqslant B$; the « internal » case is very similar. Then we can use the expansion

$$\frac{1}{R} = \frac{1}{r}\sum_{l,m}\left(\frac{4\pi}{2l+1}\right)\left(\frac{r'}{r}\right)^l Y_l^m(\Omega')^*\, Y_l^m(\Omega)\,, \qquad (r' < r), \tag{A.2}$$

where $\Omega \equiv \theta, \varphi$; $\Omega' \equiv \theta', \varphi'$ and $*$ is complex conjugation. In (A.1) the volume element $\mathrm{d}^3x' = r'^2\mathrm{d}r' \sin\theta'\mathrm{d}\theta'\mathrm{d}\varphi' \equiv r'^2\mathrm{d}r'\mathrm{d}\Omega'$ naturally involves these *inertial* polar coordinates. Substituting the expansion (A.2) into (A.1) and doing the angular integration first, we get, on account of the orthogonality relation

$$\int \mathrm{d}\Omega'\, Y_l^{m+\sigma}(\Omega')\, Y_{l'}^{m'}(\Omega')^* = \delta_{ll'}\,\delta_{m+\sigma,m'}\,, \tag{A.3}$$

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the result

$$(A.4)\qquad (A_l^m)_\sigma = \left(\frac{4\pi}{2l+1}\right)(S_l^m)_\sigma r^{-(l+1)} Y_l^{m+\sigma}(\Omega)\,. \qquad (r \geqslant B)\,.$$

Here $(S_l^m)_\sigma$ are the radial integrals given by (2.10). We shall now need the θ and φ polar components of $\boldsymbol{A}_l^m$, which are

$$(A.5)\quad \begin{cases} (A_l^m)_\theta = \dfrac{1}{2}\left(\dfrac{4\pi}{2l+1}\right) r^{-(l+1)}(\cos\theta \exp[-i\varphi](S_l^m)_+ Y_l^{m+1} + \\ \qquad\qquad + \cos\theta\exp[i\varphi](S_l^m)_- Y_l^{m-1} - 2\sin\theta(S_l^m)_0 Y_l^m)\,; \\ (A_l^m)_\varphi = \dfrac{i}{2}\left(\dfrac{4\pi}{2l+1}\right) r^{-(l+1)}(-\exp[-i\varphi](S_l^m)_+ Y_l^{m+1} + \exp[i\varphi](S_l^m)_- Y_l^{m-1})\,. \end{cases}$$

Then for the radial component of the external magnetic field

$$(A.6)\qquad B_r = (\nabla\times \boldsymbol{A})_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\varphi) - \frac{\partial}{\partial\varphi}A_\theta\right],$$

we get at first

$$(A.7)\qquad (B_l^m)_r = \frac{i}{2}\left(\frac{4\pi}{2l+1}\right)\frac{r^{-(l+2)}}{\sin\theta}\Big[-\cos\theta\exp[-i\varphi](S_l^m)_+ Y_l^{m+1} +$$
$$+\cos\theta\exp[i\varphi](S_l^m)_- Y_l^{m-1} - \sin\theta\exp[-i\varphi](S_l^m)_+\frac{\partial}{\partial\theta}Y_l^{m+1} +$$
$$+\sin\theta\exp[i\varphi](S_l^m)_-\frac{\partial}{\partial\theta}Y_l^{m-1} - m\cos\theta\exp[-i\varphi](S_l^m)_+ Y_l^{m+1} -$$
$$-m\cos\theta\exp[i\varphi](S_l^m)_- Y_l^{m-1} + 2m\sin\theta(S_l^m)_0 Y_l^m\Big]\,.$$

Manipulating this expression using the convenient differentiation formulas,

$$(A.8)\quad \begin{cases} \dfrac{\partial}{\partial\theta}Y_l^{m+1} + (m+1)\,\mathrm{ctg}\,\theta\, Y_l^{m+1} = -\sqrt{(l-m)(l+m+1)}\exp[i\varphi]\,Y_l^m\,, \\ \dfrac{\partial}{\partial\theta}Y_l^{m-1} - (m-1)\,\mathrm{ctg}\,\theta\, Y_l^{m-1} = \sqrt{(l+m)(l-m+1)}\exp[-i\varphi]\,Y_l^m\,, \end{cases}$$

we obtain the simple expression

$$(A.9)\qquad (B_l^m)_r = i\left(\frac{2\pi}{2l+1}\right) r^{-(l+2)} Y_l^m\,[\sqrt{(l-m)(l+m+1)}\,(S_l^m)_+ +$$
$$+\sqrt{(l+m)(l-m+1)}\,(S_l^m)_- + 2m(S_l^m)_0]\,.$$

Then finally

$$(U^{\text{ext}})_l^m = -\int_{\infty}^{r} (B_l^m)_r \, \mathrm{d}r \,, \qquad (r \geqslant B), \tag{A.10}$$

which yields (2.9) and completes the proof. The « internal » formula (2.8) is proved essentially the same way, using instead of (A.2) the expansion

$$\frac{1}{R} = \frac{1}{r'} \sum_{l,m} \left(\frac{4\pi}{2l+1}\right)\left(\frac{r}{r'}\right)^l Y_l^m(\Omega')^* \, Y_l^m(\Omega) \,, \qquad (r' > r). \tag{A.11}$$

RIASSUNTO (*)

Si spiega lo spostamento dell'equatore dei raggi cosmici di 45° verso ovest rispetto all'equatore geomagnetico assumendo che esistano strati atmosferici ionizzati ruotanti più lentamente della terra. Il campo magnetico risultante all'esterno di detti strati è allora un campo di dipolo il cui polo magnetico nord si trova ad ovest del comune polo geomagnetico nord. La condizione affinchè questo spostamento longitudinale sia di 45° determina $\Delta\omega \equiv \omega' - \omega$ come funzione di certi momenti radiali della conduttività dello strato (ω, ω' = velocità di rotazione dello strato e della terra, rispettivamente). L'effetto si annullerebbe o se $\Delta\omega = 0$, o se l'asse del dipolo intrinseco della terra non fosse inclinato.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XII, N. 4 16 Maggio 1959

The Fluctuations of Intensity of an Extended Light Source.

L. JÁNOSSY

Central Research Institute of Physics - Budapest

(ricevuto l'11 Gennaio 1959)

Summary. — Extending calculations carried out before (3), the present paper is dealing with the exact statistics of the fluctuation of wave trains emitted by a source the extension of which cannot be neglected. As before the individual wave trains are supposed to decay exponentially, however, it is assumed here that the individual trains have different frequencies, intensities and polarization. Various experimentally observed effects, among them that obtained with the stellar interferometer by HANBURY BROWN and TWISS (1) can be quantitatively accounted for.

1. – A beam emitted by any light source shows fluctuations of intensity; the fluctuations are caused by the random superpositions of the wave trains emitted by the individual atoms of the source. This problem was dealt with phenomenologically by HANBURY BROWN and TWISS (1) and others; the quantum mechanical treatment was considered recently by MANDEL (2) and we have also dealt with this problem previously (3). However, so as to predict quantitatively effects caused by this fluctuation it is necessary to consider the problem in more detail.

In the first part of this paper we determine the simultaneous distribution functions of the electric vectors produced by the source at given times in given points of the receiver and their time derivatives. In the second part we shall determine certain experimentally observable quantities and discuss a number of effects.

(1) R. HANBURY BROWN and R. Q. TWISS: *Proc. Roy. Soc.*, A **242**, 300 (1957).
(2) L. MANDEL: *Proc. Roy. Soc.*, **72**, 1037 (1958).
(3) L. JÁNOSSY: *Nuovo Cimento*, **6**, 111 (1957).

2. – Consider a light source situated around a point P and a receiver situated around a point Q, we denote

$$\overrightarrow{PQ} = \boldsymbol{L} \qquad \text{and} \qquad L = |\boldsymbol{L}|\,.$$

The atoms of the source are situated in points $\boldsymbol{P}_i$, while points of the receiver may be denoted Q_k, we shall write

$$\overrightarrow{PP_i} = \boldsymbol{R}_i\,, \qquad \overrightarrow{QQ_k} = \boldsymbol{r}_k\,.$$

Thus the vector pointing from P_i to Q_k is given by

$$\boldsymbol{L}_{ik} = \boldsymbol{L} + \boldsymbol{r}_k - \boldsymbol{R}_i\,, \qquad L_{ik} = |\boldsymbol{L}_{ik}|\,. \tag{1}$$

We shall suppose that the atom in P_i is suddenly excited at an instant T_i and thus starts to emit an exponentially decaying wave band. The front of the wave band arrives at a time $T_i + L_{ik}/c$ in the point Q_k. The electric vector of the wave train in Q_k at a time t_k can be written

$$\boldsymbol{E}_i^{(k)} = \boldsymbol{E}_i e(\gamma t_{ik}) \cos(\omega_i t_{ik} + 2\pi\varphi_i) \tag{2}$$

with

$$t_{ik} = T_i - t_k + L_{ik}/c\,,$$

where γ is the damping constant of the emitting atom, ω_i the frequency emitted, φ_i the phase. $\boldsymbol{E}_i$ is a vector giving polarization and intensity of the emission, finally

$$e(x) = \begin{cases} e^{-x}\,, & x > 0\,, \\ 0\,, & x < 0\,. \end{cases} \tag{3}$$

Strictly speaking, the vector $\boldsymbol{E}_i$ depends also on the position of the point Q_k. We shall, however, assume that both the light source and the receiver have dimensions small as compared with L, thus we assume

$$R_i\,,\ r_k \ll L\,, \tag{4}$$

and therefore we shall neglect in our calculation the dependence of $\boldsymbol{E}_i$ upon the position of the point Q_k. Further we shall use such an approximation that we may assume $\boldsymbol{E}_i$ to be perpendicular to $\boldsymbol{L}$; in order to investigate effects of polarization, we fix two directions « 1 » and « 2 » perpendicular to each other and perpendicular to $\boldsymbol{L}$; the components of $\boldsymbol{E}_i$ in these directions may be denoted by $E_i^{(l)}$, $l = 1, 2$. The corresponding components of the field strength

in Q_k at t_k can be written

$$E_i^{(kl)} = E_i^{(l)} e(\gamma t_{ik}) \cos(\omega_i t_{ik} + 2\pi\varphi_i^{(l)}) , \qquad l = 1, 2.$$

If $\varphi_i^{(1)} \neq \varphi_i^{(2)}$, the radiation emitted is elliptically polarized.

Finally, in order to be able to work out *intensities* of the radiation, it is useful to introduce quantities which are proportional to the time derivatives of the field strength. We shall write

$$E_i^{(klm)} = E_i^{(l)} e(\gamma t_{ik}) c_m(\omega_i t_{ik} + 2\pi\varphi_i^{(l)}) , \qquad l = 1, 2, \tag{5}$$

where we suppose

$$c_m(x) = \begin{cases} \cos x , & m = 1 , \\ \sin x , & m = 2 . \end{cases}$$

The four quantities $E_i^{(klm)}$, $l, m = 1, 2$ fully characterize the effect of the emission of the atom in P_i in the point Q_k at t_k. The total field strength in Q_k at t_k is given by the four quantities

$$E^{(klm)} = \sum_i E_i^{(klm)} , \qquad l, m = 1, 2. \tag{6}$$

We shall be interested in the simultaneous values of the field strength in two points say Q_1 and Q_2 at times t_1 and t_2, respectively. The state is described thus by eight quantities $E^{(\alpha)}$, where we suppose that α can take eight values corresponding to the eight values of the triple index klm with $k, l, m = 1, 2$. We shall also denote these eight components by one symbol

$$\mathfrak{E} = E^{(\mathrm{I})},\ E^{(\mathrm{II})},\ \dots,\ E^{(\mathrm{VIII})}, \tag{7}$$

where I, II, ..., VIII stand for 111, 112, ..., 222, respectively.

3. – Presently we determine the simultaneous probability distribution of the eight components of $\mathfrak{E}$. We denote this distribution by $P(\mathfrak{E})$; we shall restrict ourselves to the determination of the logarithmic generating function of $P(\mathfrak{E})$, thus

$$H(\mathfrak{v}) = \ln \int \exp(\mathfrak{E}\mathfrak{v}) P(\mathfrak{E})\, d\mathfrak{E} , \tag{8}$$

where the integral is an eightfold integral of the eight components of $\mathfrak{E}$ and

$$\mathfrak{E}\mathfrak{v} = \sum_{\alpha=\mathrm{I}}^{\mathrm{VIII}} E^{(\alpha)} v_\alpha ;$$

$$\mathfrak{v} = v_{\mathrm{I}},\ v_{\mathrm{II}},\ \dots,\ v_{\mathrm{VIII}}$$

are the eight transformation parameters corresponding to the components of $\mathfrak{E}$.

So as to determine $P(\mathfrak{E})$ or $H(\mathfrak{v})$ we have to make assumptions about the emissions. Each emission can be characterized by the following parameters

$$\boldsymbol{R}\,,\ \ \boldsymbol{E}\,,\ \ T\,,\ \ \omega\,,\ \ \varphi^{(1)},\ \ \varphi^{(2)},$$

namely, point and time of emission, amplitude and phases of emission. It will be convenient to split $\boldsymbol{R}$ into two components

$$\boldsymbol{R} = \boldsymbol{A} + \boldsymbol{B}\,, \tag{9}$$

where $\boldsymbol{A}$ is parallel to $\boldsymbol{L}$, while $\boldsymbol{B}$ is perpendicular to $\boldsymbol{L}$; furthermore, we are interested in the components $E^{(1)}$ and $E^{(2)}$ of $\boldsymbol{E}$. We characterize thus an emission more precisely by a set of parameters , which we denote by a symbol $\mathfrak{A}$, namely

$$\mathfrak{A} = T,\ A,\ \boldsymbol{B},\ E^{(1)},\ E^{(2)},\ \omega,\ \varphi^{(1)},\ \varphi^{(2)}\,. \tag{10}$$

We suppose the probability of an emission inside an interval $\mathfrak{A}, \mathfrak{A} + d\mathfrak{A}$ to be equal to

$$\overline{p}(\mathfrak{A})\, d\mathfrak{A}\,.$$

The probability density $\overline{p}(\mathfrak{A})$ shall be assumed not to depend explicitly either on the time T or on the average phase

$$\varphi = \tfrac{1}{2}(\varphi^{(1)} + \varphi^{(2)})\,.$$

Thus we may write

$$\overline{p}(\mathfrak{A})\, \mathrm{d}\mathfrak{A} = N p(\mathfrak{a})\, \mathrm{d}\mathfrak{a}\, \mathrm{d}T\, \mathrm{d}\varphi\,, \tag{11}$$

where the symbol $\mathfrak{a}$ stands for the parameters

$$\mathfrak{a} = A,\ \boldsymbol{B},\ E^{(1)},\ E^{(2)},\ \omega,\ \psi \tag{12}$$

only, and where we have introduced

$$\psi = \varphi^{(2)} - \varphi^{(1)}\,.$$

N is the number of impulses emitted per unit time. The quantity

$$n = N/2\gamma \tag{13}$$

has the dimension of a pure number and can be regarded as a measure of the overlap of the exponentially decreasing wave bands. In most practical cases we may suppose

$$n \gg 1 .$$

So as to determine the generating function $H(\mathfrak{v})$ we have to introduce the functions

$$\mathfrak{E}(\mathfrak{A}) = E^{(\mathrm{I})}(\mathfrak{A}),\ E^{(\mathrm{II})}(\mathfrak{A}),\ \ldots,\ E^{(\mathrm{VIII})}(\mathfrak{A}) , \tag{14}$$

with

$$E^{(\alpha)}(\mathfrak{A}) = E^{(l)} e(\gamma t_{\mathfrak{A}k}) c_m(\omega t_{\mathfrak{A}k} + 2\pi\varphi^{(l)}) \tag{15}$$

and

$$t_{\mathfrak{A}k} = T - t_k + |\boldsymbol{L} - \boldsymbol{R} + \boldsymbol{r}_k|/c . \tag{16}$$

$E^{(\alpha)}(\mathfrak{A})$ gives the eight components of the field strength which arise in Q_1 and Q_2 at the times t_1 and t_2, provided an emission took place with the parameters $\mathfrak{A}$ inside the source.

As it was shown elsewhere (4) the generating function $H(\mathfrak{v})$ can be written as

$$H(\mathfrak{v}) = \int (\exp \mathfrak{v}\,\mathfrak{E}(\mathfrak{A}) - 1)\,\overline{p}(\mathfrak{A})\,\mathrm{d}\mathfrak{A} , \tag{17}$$

where we have put

$$\mathfrak{v}\mathfrak{E}(\mathfrak{A}) = \sum_{\alpha=\mathrm{I}}^{\mathrm{VIII}} v_\alpha E^{(\alpha)}(\mathfrak{A}) .$$

4. – With the help of the generating function (17) we can determine the moments of the distribution $P(\mathfrak{E})$. We shall denote the derivatives of $H(\mathfrak{v})$ into v_α, v_β, ..., v_ε at the point $\mathfrak{v} = 0$ by H with suitable suffixes. Thus we write

$$\left(\frac{\partial H(\mathfrak{v})}{\partial v_\alpha}\right)_{\mathfrak{v}=0} = H_\alpha , \qquad \left(\frac{\partial^2 H(\mathfrak{v})}{\partial v_\alpha\,\partial v_\beta}\right)_{\mathfrak{v}=0} = H_{\alpha\beta} , \ \ldots . \tag{18}$$

Differentiating (17) into v_α, v_β, ..., v_ε we find for $\mathfrak{v} = 0$

$$H_{\alpha\beta\ldots\varepsilon} = \int E^{(\alpha)}(\mathfrak{A}) E^{(\beta)}(\mathfrak{A}) \ldots E^{(\varepsilon)}(\mathfrak{A})\,\overline{p}(\mathfrak{A})\,\mathrm{d}\mathfrak{A} . \tag{19}$$

Introducing the explicit expressions for $E^{(\alpha)}(\mathfrak{A})$ from (15) we find with the help of (11) that on account of the averaging over the phase φ

$$H_\alpha = 0 , \qquad \alpha = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{VIII}; \tag{20}$$

(4) G. Graff and L. Jánossy: in press. (*Acta Phys. Hung.* **10**, n. 3).

similarly, all odd order derivatives of H vanish for $\mathfrak{v}=0$, in particular

$$H_{\alpha\beta\gamma}=0\,. \tag{21}$$

Differentiating (8) one to four times, we find with the help of (20) the following expressions, which will be used further below

$$\left\{\begin{aligned} &\langle E^{(\alpha)}\rangle = H_\alpha = 0\,, \qquad \langle E^{(\alpha)}E^{(\beta)}\rangle = H_{\alpha\beta}\,,\\ &\langle E^{(\alpha)^2}E^{(\beta)^2}\rangle - \langle E^{(\alpha)^2}\rangle\langle E^{(\beta)^2}\rangle = H_{\alpha\alpha\beta\beta}+2H^2_{\alpha\beta}\,, \end{aligned}\right. \tag{22}$$

the meaning of the suffixes α, β is as follows:

$$\alpha = klm\,, \qquad \beta = KLM\,, \qquad k,\,l,\,m,\,K,\,L,\,M=1,\,2\,. \tag{23}$$

The expressions (22) can be evaluated using (19) and (15). We find with the help of (11) and (13) when we carry out the integration into T

$$H_{\alpha\beta} = n\int \exp\left[-\gamma\,|t_{\mathfrak{A}K}-t_{\mathfrak{A}k}|\right] c_m(\omega t_{\mathfrak{A}k}+2\pi\varphi^{(l)})\cdot \\ \cdot c_M(\omega t_{\mathfrak{A}K}+2\pi\varphi^{(L)})\,E^{(l)}E^{(L)}p(\mathfrak{a})\,\mathrm{d}\mathfrak{a}\,\mathrm{d}\varphi\,. \tag{24}$$

The above expression can be simplified if we put $m=M$ and sum over this index; we find

$$\sum_{m=M=1}^{2} H_{\alpha\beta} = n\int \exp\left[-\gamma\,|t_{\mathfrak{A}K}-t_{\mathfrak{A}k}|\right]\cdot \\ \cdot\cos\left[\omega(t_{\mathfrak{A}K}-t_{\mathfrak{A}k})+2\pi(\varphi^{(L)}-\varphi^{(l)})\right]E^{(l)}E^{(L)}p(\mathfrak{a})\,\mathrm{d}\mathfrak{a}\,. \tag{25}$$

Another important expression derived from (24) is the following:

$$2\sum_{m,\,M=1}^{2} H^2_{\alpha\beta} = n^2\int \exp\left[-\gamma(|t_{\mathfrak{A}K}-t_{\mathfrak{A}k}|+|t_{\mathfrak{A}'K}-t_{\mathfrak{A}'k}|)\right]\cdot \\ \cdot\cos\left[\omega(t_{\mathfrak{A}K}-t_{\mathfrak{A}k})-\omega'(t_{\mathfrak{A}'K}-t_{\mathfrak{A}'k})+2\pi(\varphi^{(L)}-\varphi^{(L)'}-\varphi^{(l)}+\varphi^{(l)'})\right]\cdot \\ \cdot E^{(l)}E^{(l)'}E^{(L)}E^{(L)'}p(\mathfrak{a})\,p(\mathfrak{a}')\,\mathrm{d}\mathfrak{a}\,\mathrm{d}\mathfrak{a}'\,. \tag{26}$$

Finally, we write down an expression containing fourth derivatives which will be needed further below

$$\sum_{m,\,M=1}^{2} H_{\alpha\alpha\beta\beta} = \frac{1}{2}\,n\int \exp\left[-2\gamma\,|t_{\mathfrak{A}K}-t_{\mathfrak{A}k}|\right]E^{(l)^2}E^{(L)^2}p(\mathfrak{a})\,\mathrm{d}\mathfrak{a}\,. \tag{27}$$

5. – The expressions (25), (26) and (27) can be further simplified if we neglect suitable small terms.

Split $\boldsymbol{R}$ into its longitudinal and transversal part according to (9) and split $\boldsymbol{r}_k$ similarly into

$$\boldsymbol{r}_k = \boldsymbol{a}_k + \boldsymbol{b}_k ,$$

where $\boldsymbol{a}_k$ is parallel, $\boldsymbol{b}_k$ perpendicular to $\boldsymbol{L}$; expanding (16) into powers of $1/L$ we find

$$t_{\mathfrak{A}k} = T - t_k + \frac{1}{c}\left(L + a_k + \frac{(\boldsymbol{b}_k - \boldsymbol{B})^2}{2L} + \text{terms of higher order}\right). \tag{28}$$

Neglecting higher orders we may put

$$t_{\mathfrak{A}K} - t_{\mathfrak{A}k} = t + \tau_{\mathfrak{A}kK} , \tag{29}$$

where

$$t = t_k - t_K + \frac{1}{c}\left(a_K - a_k + \frac{\boldsymbol{b}_K^2 - \boldsymbol{b}_k^2}{2L}\right), \tag{30}$$

and

$$\tau_{\mathfrak{A}kK} = \frac{\boldsymbol{B}(\boldsymbol{b}_K - \boldsymbol{b}_k)}{Lc} . \tag{31}$$

We note that

$$\gamma\tau_{\mathfrak{A}kK} = \frac{\boldsymbol{B}(\boldsymbol{b}_K - \boldsymbol{b}_k)}{L\Lambda} \sim \frac{B_0 b_0}{L\Lambda} ,$$

where

$$\Lambda = c/\gamma$$

is the half length of the individual wave trains. If we suppose

$$L,\ \Lambda \gg B_0,\ b_0 , \tag{32}$$

B_0, b_0 are the orders of magnitude of the transversal dimensions of cathode and light source, then we have

$$\gamma\tau_{\mathfrak{A}kK} \sim 0 , \tag{33}$$

and we may put

$$\gamma\,|t_{\mathfrak{A}K} - t_{\mathfrak{A}k}| \sim \gamma t . \tag{34}$$

When evaluating the integrals in Sect. **4**, we cannot neglect, however,

$$\omega\tau_{\mathfrak{A}kK} = 2\pi\,\frac{\boldsymbol{B}(\boldsymbol{b}_K - \boldsymbol{b}_k)}{L\lambda} \sim \frac{2\pi B_0 b_0}{L\lambda_0} , \tag{35}$$

where we have put $2\pi c/\omega = \lambda$, $2\pi c/\omega_0 = \lambda_0$. The latter quantity has a signi-

ficance well known in interference optics. If

$$\frac{B_0 b_0}{L\lambda_0} \ll 1 ,$$

then the source P with transversal dimensions B_0 produces on the screen with transversal dimension b_0 a coherent image.

We can, however, neglect

$$(\omega - \omega_0)\tau_{\mathfrak{A}kK} \sim \frac{2\pi B_0 b_0}{L\Lambda'} \sim 0 , \tag{36}$$

where

$$\Lambda' = \frac{2\pi c}{\Delta\omega_0} \tag{37}$$

is the coherence length of the beam as determined by the spectral width of the emissions only, when disregarding the effects of damping.

Equations (33) and (34) express that we suppose the transversal dimensions B_0, b_0 of light source and receiver both to be small as compared with

$$\Lambda = \frac{c}{\gamma} \qquad \text{and} \qquad \Lambda' = \frac{2\pi c}{\Delta\omega_0} .$$

So as to simplify the integrals in Sect. **4** it is also useful to assume some symmetry properties. If we assume the projection of the light source onto a plane perpendicular to $\boldsymbol{L}$ to have circular symmetry, we can assume the probability of an emission corresponding to a co-ordinate vector $\boldsymbol{B}$ to be equal to that with a co-ordinate vector $-\boldsymbol{B}$; therefore a value of $\tau_{\mathfrak{A}kK} = \tau$ appears with the same probability as $\tau_{\mathfrak{A}kK} = -\tau$ and under the integral we may replace

$$\cos \omega(t + \tau_{\mathfrak{A}kK})$$

by

$$\tfrac{1}{2}(\cos \omega(t + \tau_{\mathfrak{A}kK}) + \cos \omega(t - \tau_{\mathfrak{A}kK})) = \cos \omega t \cos \omega\tau_{\mathfrak{A}kK} .$$

Furthermore, we can write because of (36)

$$\omega\tau_{\mathfrak{A}kK} \sim \omega_0\tau_{\mathfrak{A}kK} .$$

Finally, if we suppose

$$\int \sin (\omega - \omega_0)t\, p(\mathfrak{a})\, d\omega \sim 0 ,$$

then we can replace under the integral

$$\cos \omega t \qquad \text{by} \qquad \cos \omega_0 t \cdot \cos (\omega - \omega_0) t \,.$$

We thus get the following approximate expressions:

$$(38) \qquad \sum_{m=M=1}^{2} H_{\alpha\beta} \approx n \exp[-\gamma |t|] \cos \omega_0 t \int E^{(l)} E^{(L)} \cos (\omega - \omega_0) t \cdot \cdot \cos 2\pi(\varphi^{(L)} - \varphi^{(l)}) \cos 2\pi \frac{\boldsymbol{B}(\boldsymbol{b}_K - \boldsymbol{b}_k)}{L\lambda_0} p(\mathfrak{a})\, d\mathfrak{a} \,,$$

$$(39) \qquad 2\sum_{m,M=1}^{2} H_{\alpha\beta}^2 = \left(n \exp[-\gamma |t|] \int E^{(l)} E^{(L)} \cos (\omega - \omega_0) t \cdot \cdot \cos 2\pi(\varphi^{(L)} - \varphi^{(l)}) \cos 2\pi \frac{\boldsymbol{B}(\boldsymbol{b}_K - \boldsymbol{b}_k)}{L\lambda_0} p(\mathfrak{a})\, d\mathfrak{a} \right)^2 .$$

(In obtaining the last expression we also supposed on grounds of symmetry

$$\int \sin 2\pi \frac{\boldsymbol{B}(\boldsymbol{b}_K - \boldsymbol{b}_k)}{L\lambda_0} d\boldsymbol{B} = 0 \,.)$$

$$(40) \qquad \sum_{m,M=1}^{2} H_{\alpha\alpha\beta\beta} = \frac{1}{2} n \exp[-\gamma |t|] \int E^{(l)2} E^{(L)2} p(\mathfrak{a})\, d\mathfrak{a} \,.$$

The expressions (38) and (39) can be further simplified if we take the distributions of A, $\boldsymbol{B}$, of ω and of $E^{(1)}$, $E^{(2)}$, $\varphi_2 - \varphi_1$ to be independent of each other

$$(41) \qquad p(\mathfrak{a}) = p_1(A, \boldsymbol{B})\, p_2(\omega)\, p_3(E^{(1)}, E^{(2)}, \varphi_2 - \varphi_1) \,;$$

we have

$$(42) \qquad \sum_{m=M=1}^{2} H_{\alpha\beta} = n \exp[-\gamma |t|] \cos \omega_0 t \langle E^{(l)} E^{(L)} \cos 2\pi(\varphi^{(L)} - \varphi^{(l)}) \rangle \cdot \cdot \langle \cos (\omega - \omega_0) t \rangle \left\langle \cos \frac{2\pi \boldsymbol{B}(\boldsymbol{b}_K - \boldsymbol{b}_k)}{L\lambda_0} \right\rangle ;$$

$$(43) \qquad 2\sum_{m,M=1}^{2} H_{\alpha\beta}^2 = \left(n \exp[-k |t|] \langle E^{(l)} E^{(L)} \cos 2\pi (\varphi^{(L)} - \varphi^{(l)}) \rangle \cdot \cdot \langle \cos (\omega - \omega_0) t \rangle \left\langle \cos \frac{2\pi \boldsymbol{B}(\boldsymbol{b}_K - \boldsymbol{b}_k)}{L\lambda_0} \right\rangle \right)^2 ;$$

$$(44) \qquad \sum_{m,M=1}^{2} H_{\alpha\alpha\beta\beta} = \frac{1}{2} n \exp[-2\gamma |t|] \langle E^{(l)2} E^{(L)2} \rangle \,.$$

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II.

We apply the expressions obtained in the previous Section to the evaluation of certain observables effects.

6. – Two-ray interferometer.

We receive in a point Q of the screen two coherent images of the source P. The effect is the same as if we were to add up the emissions received in two points Q_1 and Q_2 at one and the same instant $t_1 = t_2 = 0$. We may specify the points Q_k by writing $a_1 = \boldsymbol{b}_1 = 0$, $a_2 = x$, $\boldsymbol{b}_2 = \boldsymbol{y}$. If we regard the component polarized into the direction l we find for the intensity in Q

$$J(Q) = \sum_{m=1}^{2} (E^{(1lm)} + E^{(2lm)})^2 \,. \tag{45}$$

The expected value of $J(Q)$ is obtained with the help of (42) and (22)

$$\langle J(Q)\rangle = 2J_0(1+\varepsilon)\,,$$

where

$$J_0 = \langle E^{(l)^2}\rangle\,,$$

is the intensity of the single beam, and

$$\epsilon = \sum_{m,M=1}^{2} \overset{*}{H}_{\alpha\beta}/J_0 = \exp\left[-X/\Lambda\right]\cos\left(2\pi X/\lambda_0\right)\cdot \\ \cdot\langle\cos\left(\omega-\omega_0\right)X/c\rangle\left\langle\cos 2\pi\frac{\boldsymbol{B}\boldsymbol{y}}{L\lambda_0}\right\rangle. \tag{46}$$

and

$$X = x + \frac{y^2}{2L}\,.$$

The second factor on the right of (46) gives the interference pattern. The first describes its extinction with increasing path difference due to the damping effect, the third term gives the extinction arising from the band width $\Delta\omega_0 = \omega - \omega_0$, the last factor gives the effect of the finite size of the source. The third, respectively fourth factor approach unity if

$$\Delta\omega_0 X/c \to 0\,, \qquad \text{resp.}\ \frac{B_0 y}{L\lambda_0} \to 0\,.$$

7. – Fluctuation of intensity in a point.

The mean square fluctuation of intensity in a point Q is given by

$$\langle(\delta J^{(1l)})^2\rangle = \langle J^{(1l)^2}\rangle - \langle J^{(1l)}\rangle^2 .$$

Further

$$J^{(1l)} = E^{(1l1)^2} + E^{(1l2)^2} .$$

Thus

$$J^{(1l)^2} = \sum_{m,\,M=1}^{2} E^{(1lm)^2} E^{(1lM)^2} ,$$

and therefore according to (22)

$$\langle(\delta J^{(1l)})^2\rangle = \sum_{m,\,M=1}^{2} H_{\alpha\alpha\beta\beta} + 2\sum_{m,\,M=1}^{2} H^2_{\alpha\beta} , \qquad \alpha = 1lm,\; \beta = 1lM .$$

Thus if we put $k = K = 1$, $l = L$ we get with the help of (43) and (44)

$$\langle(\delta J^{(1l)})^2\rangle = \tfrac{1}{2} n \langle E^{(l)^4}\rangle + n^2 \langle E^{(l)^2}\rangle^2 .$$

Writing

$$n\langle E^{(l)^2}\rangle = J_0 ,$$

we have

(47) $$\langle(\delta J^{(1l)})^2\rangle / J_0^2 = 1 + \frac{\sigma^2}{2n} = \varkappa_0^2 ,$$

where

(48) $$\sigma^2 = \langle E^{(l)^4}\rangle / \langle E^{(l)^2}\rangle^2 \sim 1 .$$

The first term of (47) gives the fluctuation caused by the interference between the independent wave trains, the second term gives the Poisson fluctuation caused by the fluctuation of the number of emission processes per unit time

8. – Fluctuation on an extended cathode.

The current received from a photocathode can be taken to be proportional to the integral of the square of the field strength over the cathode surface; if we consider fluctuations of this current, we have to take into account that the electric recording instrument averages the current intensity over some period τ of time. Thus the intensity fluctuation is characterized by $\varkappa$, where

(49) $$\varkappa^2 = \left[\left\langle\left(\int_0^\tau \mathrm{d}t \int J(\boldsymbol{r}, t)\,\mathrm{d}\boldsymbol{b}\right)^2\right\rangle - \left\langle\int_0^\tau \mathrm{d}t \int J(\boldsymbol{r}, t)\,\mathrm{d}\boldsymbol{b}\right\rangle^2\right] \frac{1}{J_0^2 S^2 \tau^2} ,$$

$J_0 = \langle J^{(l)} \rangle$ being the average value of intensity and $S = \int \mathrm{d}\boldsymbol{b}$ is the illuminated part of the surface of the cathode.

Instead of (49) we may also write

$$(50) \qquad \varkappa^2 = \int_0^\tau \int_0^\tau \mathrm{d}t' \, \mathrm{d}t'' \iint \langle \delta J(\boldsymbol{r}, t') \, \delta J(\boldsymbol{r}'', t'') \rangle \, \mathrm{d}\boldsymbol{b}' \, \mathrm{d}\boldsymbol{b}'' / S^2 J_0^2 \tau^2 .$$

We note that we may put according to (22)

$$\langle \delta J^{(kl)} \, \delta J^{(Kl)} \rangle = \sum_{m,\, M=1}^{2} (H_{\alpha\alpha\beta\beta} + 2H_{\alpha\beta}^2) , \qquad \alpha = klm, \ \beta = KlM .$$

With the help of (44) and (43) we find if we assume $t = t' - t''$, $l = L$

$$(51) \qquad \langle \delta J(\boldsymbol{r}, t') \, \delta J(\boldsymbol{r}, t'') \rangle = \frac{1}{2} n \exp [-2\gamma |t' - t''|] \langle E^{(l)4} \rangle +$$
$$+ \left(n \exp [-\gamma |t' - t''| \langle E^{(l)2} \rangle \langle \cos (\omega - \omega_0)(t' - t'') \rangle \left\langle \cos 2\pi \frac{\boldsymbol{B}(\boldsymbol{b}' - \boldsymbol{b}'')}{L\lambda_0} \right\rangle \right)^2 .$$

Integrating into $\boldsymbol{b}'$ and $\boldsymbol{b}''$ we get a form factor

$$(52) \qquad g^2 = \iint \left\langle \cos 2\pi \frac{\boldsymbol{B}(\boldsymbol{b}' - \boldsymbol{b}'')}{L\lambda_0} \right\rangle^2 \mathrm{d}\boldsymbol{b}' \, \mathrm{d}\boldsymbol{b}'' / S^2 .$$

If the light spot on the cathode is coherent, *i.e.* if

$$\frac{B_0 b_0}{L\lambda_0} \ll 1 ,$$

then $g \sim 1$.

The integration into t' and t'' can be carried out when we express the square of the expectation value of $\cos (\omega - \omega_0)(t' - t'')$ by a double integral, and carry out the integration into t' and t'' first. We find

$$\varkappa^2 = g^2 f^2 + \frac{\sigma^2}{2n} \theta^2 ,$$

where we have taken

$$(53) \qquad \left\{ \begin{aligned} \theta^2 &= \frac{1}{\tau^2} \int_0^\tau \int_0^\tau \exp [-2\gamma |t' - t''|] \mathrm{d}t' \, \mathrm{d}t'' , \\ f^2 &= \frac{1}{\tau^2} \int \exp [-2\gamma |t' - t''|] \cos (\omega' - \omega_0)(t' - t'') \cos (\omega'' - \omega_0)(t' - t'') \cdot \\ &\qquad \cdot p_2(\omega') p_2(\omega'') \, \mathrm{d}\omega' \, \mathrm{d}\omega'' \, \mathrm{d}t' \, \mathrm{d}t'' ; \end{aligned} \right.$$

if $\gamma\tau \gg 1$, we find

$$\theta^2 \sim \begin{cases} 1\,, & \text{if} \quad \gamma\tau \ll 1\,, \\ 1/\gamma\tau\,, & \text{if} \quad \gamma\tau \gg 1 \end{cases}$$

and

$$f^2 \approx \int \frac{\gamma^2(\gamma^2+u^2+v^2)}{(\gamma^2+u^2+v^2)^2-4u^2v^2}\, p_2(\omega_0+u)\, p_2(\omega_0+v)\, \mathrm{d}u\, \mathrm{d}v\,. \tag{54}$$

The latter integrals can be approximately evaluated for certain extreme cases and we find

$$f^2 \sim \begin{cases} 1\,, & \text{if} \quad \gamma \gg \Delta\omega_0\,, \\ \pi p_2(\omega_0)\gamma\,, & \text{if} \quad \gamma \ll \Delta\omega_0\,. \end{cases} \tag{55}$$

Supposing

$$p_2(\omega_0) \approx \sqrt{\frac{2}{\pi}}\,\frac{1}{\Delta\omega_0}$$

(the latter relation holding exactly if $p_2(\omega)$ is a Gaussian distribution), we find

$$\pi p_2(\omega_0) \sim \frac{\sqrt{2\pi}}{\Delta\omega_0}\,, \tag{56}$$

thus

$$\varkappa^2 \approx \begin{cases} 2g^2f^2 - \dfrac{\sigma^2}{2n}\,, & \text{if } \gamma\tau,\ \ \Delta\omega_0\tau \ll 1\,, \\[2ex] \dfrac{g^2}{\gamma\tau} + \dfrac{\sigma^2}{2n\gamma\tau}\,, & \text{if } \gamma\tau \gg \Delta\omega_0\tau\,,\ 1\,, \\[2ex] \dfrac{\sqrt{2\pi}g^2}{\Delta\omega_0} + \dfrac{\sigma^2}{2n\gamma\tau}\,, & \text{if } \Delta\omega_0\tau \gg \gamma\tau\,,\ 1\,. \end{cases} \tag{57}$$

We note that according to (13)

$$2n\gamma\tau = N\tau$$

is the number of emission processes taking place during the collecting time τ; thus the second term in (57) represents the contribution to the fluctuation of the random fluctuations of the number of emission processes taking place during the time τ. The first term gives the fluctuation caused by the interference of the wave trains superposed at random. In general, the first term is more important than the second; however, if either the geometry of the arrangement is such that g^2 becomes small, or if the wave hand is broad so

that $\Delta\omega_0 \gg \gamma$, then the first term becomes small as compared with the second one which is independent of the geometry or width of the wave band.

The effect corresponding to the first term was observed recently by BRANNEN, FERGUSON and WEHLAU ([5]) counting individual photons.

9. – Simultaneous fluctuation on two cathodes.

The correlation coefficient of the fluctuations of intensity as observed on two cathodes can be obtained as

$$\Gamma_{12} = \int \langle \delta J(\boldsymbol{r}_1, t')\, \delta J(\boldsymbol{r}_2, t')\rangle \, d\boldsymbol{b}_1 \, d\boldsymbol{b}_2 \, dt' \, dt'' / (\langle \delta J_1^2 \rangle \langle \delta J_2^2 \rangle)^{\frac{1}{2}} S^2 \tau^2 , \tag{58}$$

where the integrations into t' and t'' have to carried out from 0 to τ and the integrations over $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ over the surfaces of the first and the second cathode. If we split a beam into two coherent components and project exactly the same part of the two beams on each cathode, then we find $\Gamma_{12} = 1$. If on the other hand we throw the same image on two similar cathodes but at different distances from the source, *i.e.* if $\boldsymbol{b}_1 = \boldsymbol{b}_2$ but $a_2 = a_1 + x$, we find with the help of (43), if we put $t = x/c$,

$$\Gamma_{12} = \exp[-2x/\Lambda] \left(\langle \cos 2\pi(\omega - \omega_0)\, x/c^2 \rangle g^2 f^2 + \frac{1}{2n}\, \sigma^2 \right) \Big/ \left(g^2 f^2 + \frac{\sigma^2}{2n} \right) . \tag{59}$$

For the sake of an example we suppose that the spectral distribution is a Gaussian distribution, *i.e.* that

$$p_2(\omega) = \frac{\exp[-2(\omega - \omega_0)^2/\Delta\omega_0^2]}{\sqrt{\pi}\Delta\omega_0/2} . \tag{60}$$

We have

$$\exp[-x/\Lambda] \langle \cos 2\pi(\omega - \omega_0) x/c \rangle = \exp\left[-x/\Lambda - \frac{x^2 (\Delta\omega_0)^2}{8c^2} \right] . \tag{61}$$

We see thus that the part of the correlation which depends on the frequency band decreases more rapidly with x than the frequency independent part. For large values of x, we have thus

$$\Gamma_{12} \approx \exp[-2x/\Lambda] \qquad \text{for} \qquad x\Delta\omega_0/c \gg 1 , \tag{62}$$

independent of the width of the spectrum. We note, however, that the correlation coefficient Γ_{12} thus obtained follows from a purely classical picture.

([5]) E. BRANNEN, H. I. S. FERGUSON and W. WEHLAU: *Can. Journ. Phys.*, **36**, 871 (1958).

10. – Stellar interferometer.

It was pointed out by TWISS and HANDBURRY BROWN [1], that the correlations of the fluctuations on two cathodes can be used similarly to a Michelson stellar interferometer for the determination of the angular size of a light source. This effect follows also from our formulae. Calculating the correlation coefficient of the intensity fluctuations for two small cathodes, so that $a_1 = a_2 = 0$; $\boldsymbol{b}_2 = \boldsymbol{b}_1 + \boldsymbol{y}$ we get

$$\Gamma_{12} = \left\langle \cos 2\pi \frac{\boldsymbol{B}\boldsymbol{y}}{L\lambda_0} \right\rangle^2 + \text{terms in } \frac{1}{n} . \tag{63}$$

Supposing the source to be a disc of radius B_0 we find

$$\Gamma_{12} = 4\left(\frac{\sin \alpha B_0}{\alpha B_0} - \frac{1 - \cos \alpha B_0}{\alpha B_0^2}\right)^2 , \qquad \alpha = 2\pi y / L\lambda_0 . \tag{64}$$

The correlation decreases with increasing distance y and with increasing angle of vision $2B_0/L$. We have neglected terms in $1/n$; the latter terms become predominant in the region, where the terms we have considered above are small.

11. – Coincidences observed with photon counters.

Suppose two parts of a beam to fall onto the cathodes of photon counters. The expected rate of coincidences registered with an arrangement of resolving time τ is given by

$$\varkappa_{12} = 2p_0^2 \int_0^\tau \langle J(\boldsymbol{r}_1, t) J(\boldsymbol{r}_2, t + t') \rangle \, \mathrm{d}\boldsymbol{b}_1 \, \mathrm{d}\boldsymbol{b}_2 \, \mathrm{d}t' , \tag{65}$$

where p_0 is the expected rate of impulses produced by the unit intensity falling on the cathode.

With the help of (43), (44), (48), (57)

$$\varkappa_{12} = 2N_1^2 \tau \left(1 + \frac{\sigma^2}{n\tau} \int_0^\tau \exp[-2\gamma t'] \, \mathrm{d}t' + \frac{g^2}{\tau} \int_0^\tau \exp[-2\gamma t'] \langle \cos(\omega - \omega_0) t' \rangle^2 \, \mathrm{d}t' \right) , \tag{66}$$

with

$$N_1 = J_0 p_0 S .$$

J_0 is the average intensity and S the surface of each cathode.

Evaluating the last integral we find for $\gamma\tau \gg 1$

$$\varkappa_{12} = 2N^2\tau\left(1 + \frac{g^2}{\gamma\tau}\left\langle\frac{1}{1+(\omega-\omega_0)^2/\gamma^2}\right\rangle + \frac{\sigma^2}{2n\gamma\tau}\right). \tag{67}$$

Further we may put

$$\left\langle\frac{1}{1+(\omega-\omega_0)^2/\gamma^2}\right\rangle \approx \begin{cases} 1, & \text{if } \Delta\omega_0 \ll \gamma, \\ \pi p_2(\omega_0)/\gamma, & \text{if } \Delta\omega_0 \gg \gamma. \end{cases} \tag{68}$$

The rate of accidental coincidences, in case of a constant intensity would be

$$\varkappa_{12}^{(0)} = 2N^2\tau. \tag{69}$$

The excess $\varkappa_{12} - \varkappa_{12}^{(0)}$ consists of two components. The first one, which is in general the more important, is sensitive to the degree of coherence of the light spot on the cathode and in addition to the spectral width of the band of emissions. The second term (which represents a kind of Poisson fluctuation) depends only on the rate of emission processes of the source. The latter term should become preponderant if the first term becomes small on account of geometry or band width.

We hope to return to the problem as to what modifications in the above formulae are to be expected if the light waves are subjected to quantization.

RIASSUNTO (*)

Estendendo calcoli eseguiti prima della ([3]) il presente lavoro si occupa della statistica esatta della fluttuazione dei treni d'onde emessi da una sorgente di estensione non trascurabile. Come precedentemente, si assume che i singoli treni d'onde decadano esponenzialmente; tuttavia qui si assume che i singoli treni abbiano differenti frequenze, intensità e polarizzazioni. Vari effetti osservati sperimentalmente, fra cui quello ottenuto coll'interferometro stellare da HANBURY BROWN e TWISS, si possono giustificare quantitativamente.

(*) *Traduzione a cura della Redazione.*

Spettrometro automatico per raggi gamma.

M. Frank

Istituto Superiore di Sanità - Roma

(ricevuto il 2 Marzo 1959)

Riassunto. — Si descrive uno spettrometro per raggi γ a singolo canale. Il numero degli impulsi contati su ciascun canale è registrato automaticamente su nastro di carta, ed al termine della registrazione l'apparato si trova predisposto per la misura del canale successivo.

La misura dello spettro completo di un radioisotopo mediante un analizzatore d'ampiezza degli impulsi a singolo canale richiede da parte dell'operatore un tempo notevole, rapidamente crescente con l'aumentare della risoluzione richiesta. È quindi particolarmente desiderabile che un'apparecchiatura destinata a questo scopo sia resa automatica.

È stato realizzato presso l'Istituto Superiore di Sanità di Roma uno spettrometro a scintillazione per raggi γ capace di effettuare automaticamente l'esplorazione dello spettro e registrare in colonna, di volta in volta, i conteggi relativi ai vari canali.

L'apparato, il cui schema a blocchi è mostrato in Fig. 1, comprende essenzialmente:

— un cristallo scintillatore di NaI(Tl);
— un fotomoltiplicatore;
— un alimentatore stabilizzato per il fotomoltiplicatore;
— un amplificatore lineare con uscita fino a 50 V;
— un analizzatore d'ampiezza degli impulsi a singolo canale;
— un registratore automatico di numeri di impulsi.

L'analizzatore d'ampiezza degli impulsi deriva, con poche modifiche di dettaglio, dalla corrispondente unità dello spettrometro descritto da M. Ageno

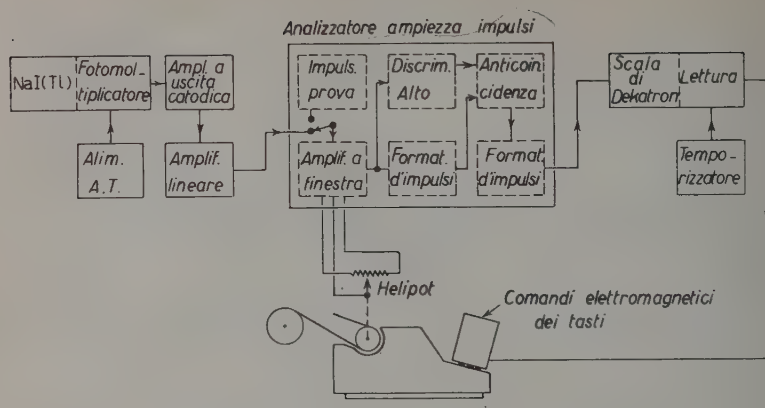


Fig. 1. – Schema a blocchi dello spettrometro automatico.

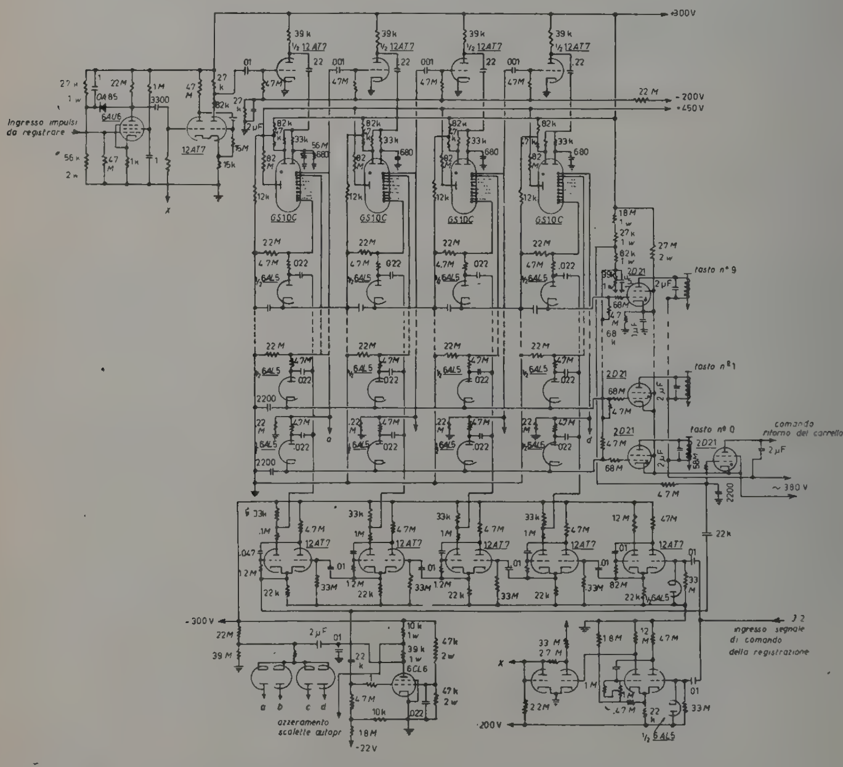


Fig. 2. – Schema del registratore.

Fig. 3. – Fotografia del potenziometro comandato dal carrello della macchina da scrivere.

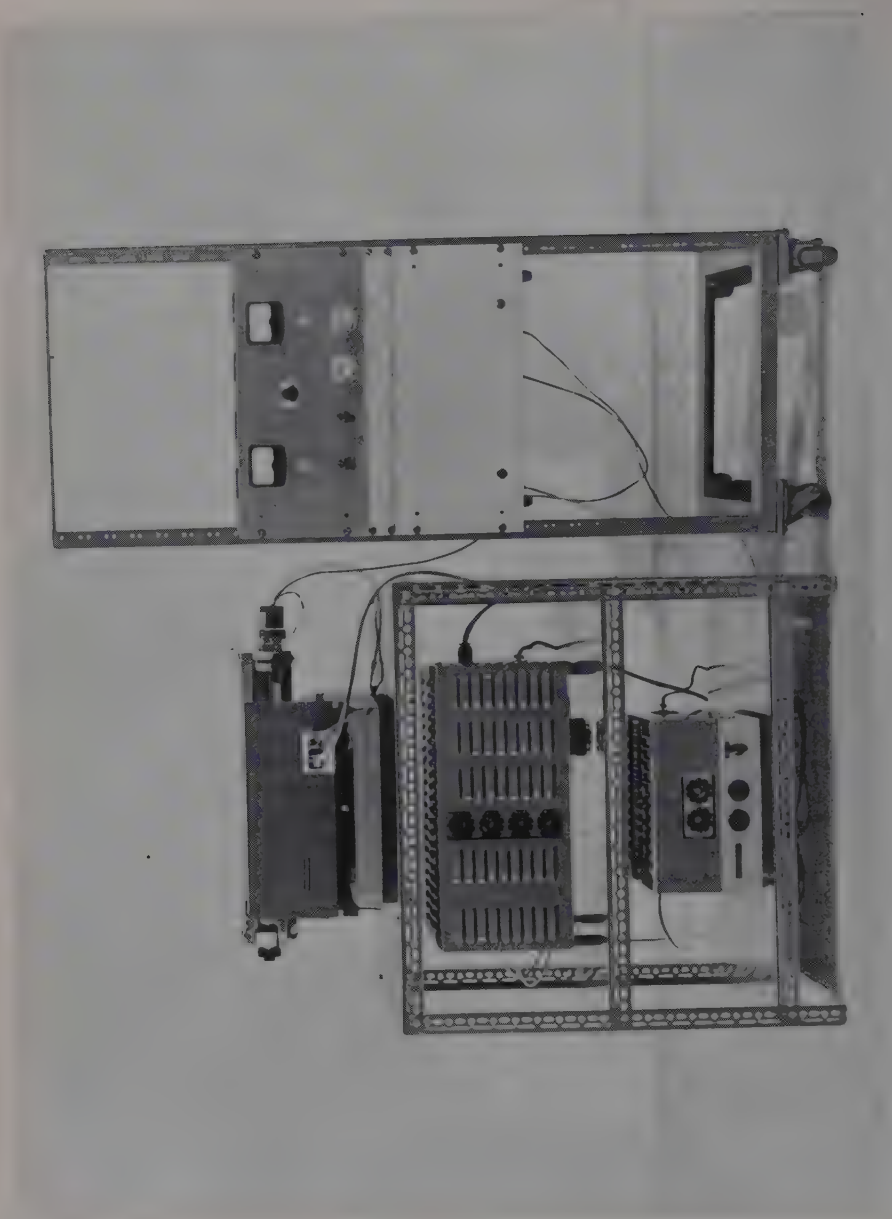


Fig. 4. – Fotografia dell'analizzatore e del registratore.

et al. (1). Anche in questo analizzatore è prevista la possibilità di spostare a mano il canale, mediante un potenziometro elicoidale a dieci giri.

Il registratore, il cui schema è riportato in Fig. 2, deriva, con una modifica concernente la sequenza delle operazioni di registrazione, dal tipo descritto (2,3) dall'autore della presente nota. Tale apparato si compone di una scala di quattro selettori Dekatron e di un circuito di lettura elettronico mediante il quale, ad intervalli di tempo stabiliti con un temporizzatore, viene sospeso il conteggio, ed il numero di impulsi contati dalla scala viene registrato su carta per mezzo di una macchina da scrivere elettrica, i tasti dei numeri della quale sono comandati per mezzo di elettromagneti. Dopo che è stata effettuata la registrazione, la scala viene riazzerata ed inizia nuovamente il conteggio. Il tempo perduto nelle operazioni di registrazione è di 1 s.

Per il funzionamento automatico dello spettrometro, in luogo del potenziometro elicoidale contenuto nell'analizzatore, viene inserito nel circuito mediante un commutatore un altro potenziometro, uguale al precedente (*) e montato sul carrello della macchina da scrivere (Fig. 3). Tale potenziometro è azionato dal movimento di rotazione del rullo. Mediante opportuno rapporto di ingranaggi tra l'asse del rullo e l'asse del potenziometro, si può ottenere la rotazione completa di 3600° dell'asse del potenziometro per una rotazione del rullo corrispondente alla scrittura di tanti numeri incolonnati quanti sono i canali in cui si desidera suddividere lo spettro (**).

Il potenziometro del carrello è supportato mediante una boccola sull'asse. Com'è visibile nella fotografia di Fig. 3, allo statore del potenziometro è fissata una camma che presenta all'estremo una rientranza nel profilo. Il nottolino fissato alla leva di un microinterruttore può premere elasticamente sulla camma in corrispondenza della rientranza, trattenendo dalla rotazione lo statore del potenziometro. Prima di iniziare la serie di misure, il rullo della macchina da scrivere viene ruotato a mano in senso inverso al normale senso di rotazione, fino a che il cursore del potenziometro, arrivato all'inizio della corsa, trascina lo statore, ed il nottolino impegna la camma. Al termine dell'esplorazione dello spettro il cursore arriva a fondo corsa ed obbliga lo statore a ruotare: si libera così la levetta del microinterruttore e quest'ultimo toglie corrente alla macchina da scrivere.

In Fig. 4 è mostrato l'analizzatore d'ampiezza degli impulsi e relativo alimentatore (entrambi montati su rack), ed il registratore costituito dalla

(1) M. Ageno, G. Cortellessa ed R. Querzoli: *Rend. Ist. Sup. Sanità*, **18**, 344 (1955).

(2) M. Frank: *Selected Scientific Papers from the Istituto Superiore di Sanità*, vol. **1**, part II (1958), p. 281.

(3) M. Frank: *Atti del Congresso Scientifico - Sezione Elettronica. V Rassegna Internazionale Elettronica-Nucleare* (1958).

(*) Le resistenze dei potenziometri possono essere portate esattamente allo stesso valore mediante shunt di correzione.

(**) Con la macchina da scrivere impiegata, di costruzione italiana, in un giro del rullo sono contenuti esattamente 26 numeri incolonnati con uno spazio, e 13 numeri con tre spazi. Con una coppia di ingranaggi con rapporto 2.6 : 1 (di 65 e 25 denti) si possono ottenere quindi, per dieci giri dell'asse del potenziometro, 100 numeri incolonnati corrispondenti a 100 canali o 50 numeri per 50 canali, a seconda della spaziatura usata.

macchina da scrivere con il complesso degli elettromagneti per il comando dei tasti, dalla unità comprendente la scala di Dekatron ed il circuito di lettura, e dal temporizzatore.

Poichè gli errori dovuti ad imprecisioni meccaniche sono minori di una parte su 10^3 per giro, e quindi — dato che gli errori non si sommano sui vari giri — di una parte su 10^4 per la intera corsa del potenziometro, il sistema di predisposizione dei successivi canali qui descritto assicura almeno altrettanta precisione di quella ottenibile con la regolazione manuale del potenziometro.

SUMMARY

A single channel γ-ray spectrometer is described. The number of pulses counted on each channel is automatically recorded on a paper ribbon, and the apparatus is present for measuring the next channel.

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

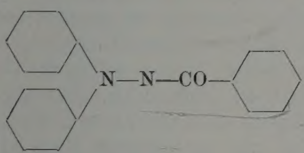
Nuclear Hyperfine Structure of a Hydrazinic Free Radical (*).

I. S. Ciccarello, T. Garofano and M. Santangelo

Istituto di Fisica dell'Università - Palermo

(ricevuto il 13 Dicembre 1958)

We have undertaken a research program on the Electronic Spin Resonance (E.S.R.) of organic free radicals. The E.S.R. of the diphenyl-benzoyl-hydrazyl free radical has been investigated in the X-band.



This radical, which cannot be isolated in the solid state ([1]), has been investigated in carbon tetrachloride solutions at $-20\,^{\circ}C$ in an inert atmosphere. Under these conditions the solutions are stable for weaks, whereas at higher temperatures they rapidly decompose. At lower temperatures the equilibrium between the free-radical form and the dimeric form gradually shifts towards the latter.

Our measurements were carried out using $5\cdot10^{-4}$ mol solutions on specimens containing approximately 10^{16} paramagnetic centers. The experimental set-up, which uses high stabilization of both the magnetic field and the microwave frequency, has been described elsewhere ([2]). The specimen was contained in a small Pyrex tube and placed at the center of a TE_{011} cylindrical cavity: changes in the power reflected by the latter were detected. The recorded derivative of the absorption curve is shown in Fig. 1 (curve 1).

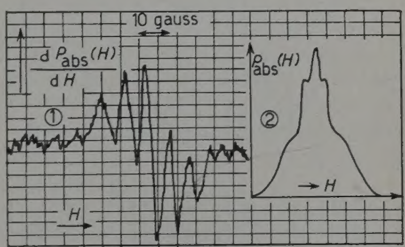


Fig. 1. – Curve 1: recorded derivative of the absorption curve (X-band-modulation: 1 gauss). Curve 2: integrated curve (from 1).

(*) This work forms part of a program of research undertaken with the joint support of the Sicilian Comitato Regionale Ricerche Nucleari and the Consiglio Nazionale delle Ricerche, for which the Authors wish to express their sincere gratitude.

([1]) S. Goldschmidt: *Ann. der Chemie*, **437**, 194 (1924).

([2]) M. B. Palma-Vittorelli, and M. U. Palma: *Supplemento al Nuovo Cimento*, **7**, 139 (1958).

The spectrum shows five resolved, equally-spaced lines having a separation of approximately 7.5 G. They are centered on a g-value very close to the value 2.0036 of D.P.P.H. [3]. This structure is due to the $I \cdot S$ coupling with the nitrogen nuclei: the presence of five lines and the ratio of their intensities show that the two nitrogen nuclei are equivalent, since if they were not they would give rise to nine lines. There was no evidence of a further separation due to the interaction with the hydrogen nuclear magnetic moments: this is consistent with a wash-out effect, which is expected because of the number of the hydrogen nuclei, and of their inequivalence in the phenylic rings. The observed spectrum is therefore consistent with the following spin-Hamiltonian:

$$(1)\quad \mathcal{H} = g\beta H \cdot S + AS \cdot (I_{N_1} + I_{N_2}) ,$$

where the subscripts N_1 and N_2 refer to the two nitrogen nuclei.

The A value is about 25% smaller than that of D.P.P.H. [4]; this shows a corresponding increase of charge transfer on the phenylic rings, presumably due to the presence of the CO group.

Further work is in progress along these lines.

* * *

We wish to thank M. B. Palma-Vittorelli and M. U. Palma for friendly and valuable discussions.

(3) N. A. Holden, C. Kittel, F. R. Merritt, W. A. Yager: *Phys. Rev.*, **77**, 147 (1950).

(4) C. A. Hutchinson, R. C. Pastor, A. G. Kowalsky: *Journ. Chem. Phys.*, **20**, 534 (1952).

LIBRI RICEVUTI E RECENSIONI

G. I. TAYLOR - Scientific Papers edited by G. K. Batchelor. Vol. I: *Mechanics of Solids*. Cambridge, University Press.

Questo volume, primo di una serie di quattro, riunisce quarantuno memorie di Sir Geoffrey Taylor limitatamente al solo campo della meccanica dei solidi, disposte in ordine cronologico dal 1917 al 1956: alcune di esse sono per la prima volta disponibili al pubblico in quanto furono redatte per amministrazioni militari o statali durante la seconda guerra.

Si distingue per il posto che ha assunto nella storia della resistenza dei materiali il gruppo di note dedicate alla teoria delle dislocazioni, con la quale G. I. TAYLOR ha interpretato i fenomeni di deformazione plastica dei materiali cristallini: scorrimento lungo i piani del reticolo e in direzione privilegiata, aumento di resistenza al crescere della deformazione secondo la curva di incrudimento.

La elaborazione teorica era stata preceduta dai classici lavori sperimentali, condotti in collaborazione con C. F. ELAM, su cristalli di alluminio, intesi a cogliere la connessione tra distorsione e assi cristallografici.

Nelle memorie del 1934 TAYLOR interpretò lo scorrimento plastico elementare come risultato finale di successivi « salti » con cui, uno dopo l'altro, gli atomi che si trovano da una banda del piano di scorrimento guadagnano un posto nel reticolo cristallino: tale successione corrisponde al migrare di una « dislocazione », come dire di una lacuna (un posto vuoto) originariamente presente nel reticolo stesso.

Ammesso che ogni atomo occupi la posizione di energia potenziale minima rispetto ai vicini e che per sfuggire da tale posizione di equilibrio stabile debba normalmente superare congrue barriere di potenziale, il TAYLOR dimostra che in corrispondenza alla dislocazione tali barriere sarebbero ridotte: l'applicazione, allora, di una tensione tangenziale esterna anche minima provocherebbe il salto dell'atomo adiacente nel posto vuoto.

D'altronde ogni dislocazione induce nel suo intorno una distribuzione di tensioni elastiche (l'A. inserisce tale effetto nel quadro delle distorsioni di Volterra). I campi elastici di due o più dislocazioni interagiscono con forze, che sono attrattive per dislocazioni di segno opposto, cosicchè per provocare la migrazione delle dislocazioni (e in definitiva lo scorrimento plastico) occorre applicare una tensione esterna finita e anzi crescente al crescere dello scorrimento.

Su tali basi TAYLOR perviene a dedurre analiticamente una curva di incrudimento parabolica in buona concordanza con le esperienze su alluminio, ferro, oro, rame.

Altri lavori rimangono nel campo della elasticità classica, dai primi in cui il TAYLOR applicò l'analisi della membrana a problemi di torsione e flessione, a quello in cui diede la soluzione rigo-

rosa della instabilità dell'equilibrio di una piastra quadrata incastrata ai lati, soggetta ad uguale compressione nelle due direzioni.

Nel campo degli studi sperimentali timane fondamentale il lavoro compiuto con H. QUINNEY su tubi di rame soggetti a tensione e torsione oltre i limiti elastici, i cui risultati vennero messi a confronto con i criteri di plasticità di Mohr e di Von Mises.

Sul criterio di plasticità e sulla curva di incrudimento generalizzata il TAYLOR è tornato più recentemente in relazione ai lavori di HILL, LEE e TUPPER.

Suggestivi problemi di plasticità applicata sono, tra gli altri, quelli risolti nella nota del 1942 sulla propagazione dell'onda plastica in un filo esteso da un carico istantaneo; in quella del 1947 sulla formazione del foro circolare prodotto da una punta conica rotante in un disco sottile, con previsione qualitativa e quantitativa dello ispessimento che si forma al bordo, nonchè della distribuzione delle tensioni nella regione plastica immediata e in quella elastica circostante.

Si aggiungano ancora gli studi di elasticità su piastre non isotrope (fogli di legno), quelli sulla propagazione delle onde elastiche in seguito ad esplosioni, e si avrà una idea della molteplicità dei risultati conseguiti dal TAYLOR e documentati in questo prezioso volume, presentato con solenne eleganza.

D. GENTILONI SILVERI

Direttore responsabile: G. POLVANI Tipografia Compositori - Bologna

Questo fascicolo è stato licenziato dai torchi il 14-V-1959